Modules over a Category and Finiteness Conditions

Transfer Thesis

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June 10, 2013

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0 Introduction

In the 1930's Hurewicz made the fundamental observation that a path-connected aspherical space is uniquely determined, up to homotopy equivalence, by its fundamental group G. Such a space is called a *model for BG* or *Eilenberg-Mac Lane space* $K(G,1)$. The universal cover $\mathrm{E}G$ of $\mathrm{B}G$ is called the classifying space for free actions, it is the terminal object in the G-homotopy category of free G-CW-complexes. One can use invariants of these spaces to study the groups themselves, for example, calculating the cohomology $H^*(\mathrm{B}G)$ gives the group cohomology $H^*(G)$. By the 1940's a purely algebraic definition of group cohomology was formulated, replacing the space $\mathrm{B}G$ with a projective resolution of $\mathbb{Z}$ by $\mathbb{Z}G$-modules.

The *geometric dimension* of a group G, denoted $\operatorname{gd} G$, is the minimal dimension of a CW-model for EG. This is a suprisingly subtle invariant, and often the easiest way to approach it is via the related algebraic invariant of cohomological dimension. The *cohomological dimension* of a group G, denoted $\operatorname{cd} G$, is the minimal length of the projective resolution of $\mathbb{Z}$ by projective $\mathbb{Z}G$. It's easy to see that $\operatorname{gd} G = 0$ if and only if $\operatorname{cd} G = 0$ if and only if G is the trivial group and by a theorem of Stallings and Swan, $\operatorname{cd} G = 1$ if and only if $\operatorname{gd} G = 1$ if and only if G is a free group [Sta68][Swa69]. Eilenberg and Ganea conjectured that $\operatorname{cd} G = \operatorname{gd} G$ for all groups and, along with Stallings and Swan's result for the dimension one case, proved this conjecture for all cases, except for the possibility that $\operatorname{cd} G = 2$ and $\operatorname{gd} G = 3$ [EG57].

A group G has type F_n if it admits a model for $\mathrm{B}G$ with finite n-skeleton, and on the algebraic side G has type FP_n if $\mathbb{Z}$ admits a projective resolution of $\mathbb{Z}G$-modules, finitely generated up to dimension n. The conditions F_1, FP_1 and finitely generated are all equivalent, but the situation is more complex for larger n. F_n implies FP_n, using the free resolution of $\mathbb{Z}G$-modules arising from the cellular chain complex of $\mathrm{E}G$. The condition F_2 is equivalent to finitely presented and FP_n with F_2 implies F_n [Bro94, §VIII.7]. Examples constructed by Bestvina and Brady show there are groups that are FP_n but neither FP_{n+1} nor finitely presented for all n [BB97].

For an overview of finiteness conditions see [Bro94, Chapter VIII], [Bie81] and [Geo08, Chapter II].

Spaces which admit free G-actions can be very difficult to find, not many occur in nature and they are often large and unwieldy. If G is a non-trivial finite group, for instance, a model for $\mathrm{E}G$ is necessarily infinite dimensional. We weaken the freeness condition, looking instead for CW-spaces which admit proper actions (where the cell stabilisers are finite subgroups of G). In analogy with the definition of a model for $\mathrm{E}G$, a *model for* $E_{\mathcal{F}in}G$ if it is a terminal object in the G-homotopy category of proper G-CW complexes.

There are many constructions of models for $\mathrm{E}_{\mathcal{F}in}G$ for different classes of groups, generic constructions which work for all classes of groups [Mil56][Seg68] and constructions, often smaller and easier to work with, for specific classes of groups, such as the Rips complex of a hyperbolic group [MS02][Lüc03]. This idea can be further generalised to the study of models for $\mathrm{E}_{\mathcal{F}}G$, terminal objects in the homotopy category of G-CW complexes with stabilisers in some family $\mathcal{F}$ of subgroups of G.

Models for $\mathrm{E}_{\mathcal{F}in}G$ and models for $\mathrm{E}_{\mathcal{VC}yc}G$, where $\mathcal{VC}yc$ denotes the family of virtually cyclic subgroups, have recently become of great interest because they appear on one side of the Baum-Connes and Farrell-Jones conjectures respectively [LR05]. These are deep conjectures which have far reaching consequences in mathematics [MV03][BLR08].

0.1 Geometric and Cohomological Dimension

We define the Bredon geometric dimension of a group G, denoted $\operatorname{gd}_{\mathcal{F}in} G$, to be the minimal dimension of a model for $\mathrm{E}_{\mathcal{F}in}G$. As in the case of ordinary geometric dimension this invariant is fairly intractable, so many algebraic invariants have been proposed to mimic this property. The most successful is the Bredon cohomological dimension $\mathcal{O}_{\mathcal{F}in} \operatorname{cd} G$. Indeed, it is easy to show that $\mathcal{O}_{\mathcal{F}in} \operatorname{cd} G \leq \operatorname{gd}_{\mathcal{F}in} G$ and Lück and Meintrup provide that $\operatorname{gd}_{\mathcal{F}in} G \leq \max\{\mathcal{O}_{\mathcal{F}in} \operatorname{cd} G, 3\}$ [LM00, Theorem 0.1]. Furthermore, Dunwoody has shown that $\mathcal{O}_{\mathcal{F}in} \operatorname{cd} G = 1$ implies that $\operatorname{gd}_{\mathcal{F}in} G = 1$ [Dun79]. This leaves open only the possibility of an Eilenberg-Ganea phenomenon, a group for which $\mathcal{O}_{\mathcal{F}in} \operatorname{cd} G = 2$ and $\operatorname{gd}_{\mathcal{F}in} G = 3$. Brady, Leary and Nucinkis show that this can indeed happen [BLN01].

The Bredon cohomological dimension is, unfortunately, also difficult to compute in practice so there has been a lot of attention given to more easily computable invariants. Perhaps the most obvious invariant when G is torsion-free is the virtual cohomological dimension. If G has finite virtual cohomological dimension, say $\operatorname{vcd} G \leq n$, then by a theorem of Serre [Ser71][Bro94, VIII.3], G has finite geometric dimension. Except in the trivial case however, Serre's theorem does not provide that $\operatorname{vcd} G = \mathcal{O}_{\mathcal{F}in} \operatorname{cd} G$ or even a good bound. The conjecture that $\operatorname{vcd} G = \mathcal{O}_{\mathcal{F}in} \operatorname{cd} G$ is known as Brown's conjecture [Bro79], and is false in general, although there are classes of groups for which it is known to hold [MPN10].

In [Kro93], Kropholler introduces a hierarchically defined class of groups $\mathbf{H}\mathcal{F}$. Let $\mathbf{H}_1\mathcal{F}$ be the class of groups acting properly on a contractible space with finite stabilisers, and then let $\mathbf{H}\mathcal{F}$ to be the smallest class containing $\mathbf{H}_1\mathcal{F}$ and with the property that if G acts on a contractible complex with stabilisers in $\mathbf{H}\mathcal{F}$ then G is in $\mathbf{H}\mathcal{F}$. Kropholler and Mislin show that if G is a member of $\mathbf{H}\mathcal{F}$ and is FP_∞ then G has finite Bredon geometric dimension [KM98]. This class $\mathbf{H}\mathcal{F}$ is very large, being closed under countable direct limits, free products with amalgamation and HNN extension, but it does not contain all groups. Thompsons group F is FP_∞ yet has infinite cohomological dimension over $\mathbb{Q}$, since it contains an infinitely generated free abelian subgroup [BG84][CFP96], thus Kropholler' s argument cannot be weakened to require only FP_∞. It's quite difficult to produce examples of groups which do not belong to $\mathbf{H}\mathcal{F}$, for a long time the only known examples contained Thompsons groups as subgroups, more recently more examples have been constructed [ABJ$^+$09][Gan12].

Kropholler's Theorem relies heavily on the fact that these $\mathbf{H}\mathcal{F}$ groups of type FP_∞ have bounded lengths of finite subgroups, and in fact almost all known results bounding the geometric dimension rely on bounded lengths of finite subgroups. Almost nothing is known in the unbounded case, but it is clear that such a bound is not required [Lüc00, Example 1.11]. Recall that the length of a finite subgroup F is defined to be the length of the longest chain of subgroups of F:

$$1 \lneq F_1 \lneq \cdots \lneq F_n = F$$

Lück's result [Lüc00, Theorem 1.10], can be viewed as an improvement of Krophollers result, he introduces a new invariant $B(d)$, which is satified by G if and only if $\operatorname{pd}_{\mathbb{Z}G} U \leq d$ for any $\mathbb{Z}G$-module U which is projective when restricted to any finite subgroup of G, and proves that the geometric dimension is bounded by $\max(3, d) + l(d+1)$, where l is the the bound on the lengths of finite subgroups of G. Martinez-Perez obtains a cohomological analog of Lück's result in [MP07], showing that if G is known to have finite Bredon cohomological dimension, it is bounded above by $l + \operatorname{pd}_{\mathbb{Z}G} B(G, \mathbb{Z})$, where l is the bound on the lengths of the finite subgroups and $B(G, \mathbb{Z})$ is the ring of bounded functions. Note that if G has $B(d)$ then $\operatorname{pd}_{\mathbb{Z}G} B(G, \mathbb{Z}) \leq d$.

By a Theorem of Bouc [Bou99] and Kropholler-Wall [KW11], if G acts properly on a contractible G-CW-complex X, the cellular chain complex C_*X splits when regarded as a complex of $\mathbb{Z}H$-modules for any finite subgroup H of G. Nucinkis introduced a cohomology theory called $\mathcal{F}$-cohomology to mimic this property in [Nuc99]. $\mathcal{F}$-cohomology can be thought of as a generalisation of cohomology relative to a single subgroup, instead taking cohomology relative to a family of subgroups $\mathcal{F}$. It's also a special case of the relative cohomology defined in [ML95, IX]. This theory gives rise to a new algebraic invariant, the $\mathcal{F}in$-cohomological dimension, denoted $\mathcal{F}in \operatorname{cd} G$. It is an open question whether $\mathcal{F}in \operatorname{cd} G < \infty$ implies $\operatorname{gd}_{\mathcal{F}in} G < \infty$, or indeed if $\mathcal{F}in \operatorname{cd} G < \infty$ implies $G \in \mathbf{H}_1\mathcal{F}$.

Mackey Functors for finite groups have been well-studied, as they provide an abstract framework with properties common to structures such as group cohomology, the representation ring, topological and algebraic K-theory [Web00], there are also some applications of Mackey functors for infinite groups[Lüc02]. In [MPN06], Martinez-Perez and Nucinkis study Mackey functors for infinite groups, constructing a cohomology theory from this in a similar way to the construction of Bredon cohomology. This creates a new invariant, the Mackey cohomological dimension, denoted $\mathcal{M}_{\mathcal{F}in} \operatorname{cd} G$. This is a lower bound for $\mathcal{O}_{\mathcal{F}in} \operatorname{cd} G$, but they prove that for all virtually torsion-free groups there is equality:

$$\operatorname{vcd} G = \mathcal{F}in \operatorname{cd} G = \mathcal{M}_{\mathcal{F}in} \operatorname{cd} G$$

This is further improved by Degrijse [Deg13b], who shows that for groups G with a bound on the orders of their finite subgroups,

$$\mathcal{F}in \operatorname{cd} G = \mathcal{M}_{\mathcal{F}in} \operatorname{cd} G$$

In the process of proving this result, Degrijse studies a specific family of Mackey functors known as cohomological Mackey functors. Considering these gives rise to a new finiteness condition - the cohomological Mackey dimension - denoted $\mathcal{H}_{\mathcal{F}} \operatorname{cd} G$. Degrijse shows that $\mathcal{F}in \operatorname{cd} G = \mathcal{H}_{\mathcal{F}} \operatorname{cd} G$ for all groups G with $\mathcal{H}_{\mathcal{F}} G < \infty$. Note that in [Deg13b], the notation $\operatorname{cd}_{\text{coMack}} G$ is used instead of $\mathcal{H}_{\mathcal{F}} \operatorname{cd}$, the reason for the symbol $\mathcal{H}$ will be explained in Section 4.

For all the algebraic invariants so far mentioned there is the following chain of inequalities [BLN01, Theorem 2][MPN06, 3.9,4.3][Deg13b]:

$$\operatorname{cd}_{\mathbb{Q}} G \leq \mathcal{F}in \operatorname{cd} G = \mathcal{H}_{\mathcal{F}} \operatorname{cd} G \leq \mathcal{M}_{\mathcal{F}in} \operatorname{cd} G \leq \mathcal{O}_{\mathcal{F}in} \operatorname{cd} G$$

It's unknown if the finiteness of any of these invariants, except the Bredon cohomological dimension, implies the Bredon cohomological dimension is finite.

In [LN03], Leary and Nucinkis use finite extensions of Bestvina-Brady groups to construct virtually torsion-free groups for which $\mathcal{O}_{\mathcal{F}in} \operatorname{cd} G = 3n$ and $\operatorname{vcd} G = \mathcal{F}in \operatorname{cd} G = 2n$ for all integers n.

In Section 1 we construct modules over a category in generality, Section 2 then specialises to the orbit category $\mathcal{O}_{\mathcal{F}}$, where we discuss the Bredon cohomological dimension $\mathcal{O}_{\mathcal{F}} \operatorname{cd}$. We also consider a condition related to $\mathcal{O}_{\mathcal{F}} \operatorname{cd}$, the covariant cohomological dimension, and completely classify those groups with covariant cohomological dimension n (see Section 2.4). Section 2.6.2 contains examples of groups where various notions of dimension differ.

There are no new results in Section 3 on the Mackey cohomological dimension, although for completeness we do provide a brief overview of known results. In Section 4.4 we show the cohomological Mackey cohomological dimension $\mathcal{H}_{\mathcal{F}in} \operatorname{cd}$ and the $\mathcal{F}$ cohomological dimension for the family of finite subgroups $\mathcal{F}in \operatorname{cd}$ always agree, a very slight improvement of [Deg13b, 6.2.16].

0.2 FP_n Conditions

Related to the geometric dimensions are the generalisations of the FP_n conditions from ordinary group cohomology. These generalise to the $\mathcal{O}_{\mathcal{F}}\mathrm{FP}_n$ conditions in Bredon cohomology, the $\mathcal{M}_{\mathcal{F}}\mathrm{FP}_n$ conditions for Mackey functors, the $\mathcal{H}_{\mathcal{F}}\,\mathrm{FP}_n$ conditions for cohomological Mackey functors, and the $\mathcal{F}$-FP_n conditions arising from $\mathcal{F}$-cohomology.

Of these the $\mathcal{O}_{\mathcal{F}}\mathrm{FP}_n$ conditions are the most studied, and in fact completely classified in the case $\mathcal{F} = \mathcal{F}in$: G is $\mathcal{O}_{\mathcal{F}in}\,\mathrm{FP}_n$ if and only if G has finitely many conjugacy classes of finite subgroups and $WK = N_G K/K$ is FP_n for all finite subgroups K of G [KMPN10, Lemmas 3.1, 3.2]. A version of the Bieri-Eckmann criterion for $\mathcal{O}_{\mathcal{F}}\mathrm{FP}_n$ is also proved in [MPN11, Section 5] (see [Bie81, Section 1.3] for the classical case).

The condition $\mathcal{F}in\,\mathrm{FP}_n$, coming from $\mathcal{F}$-cohomology, is much less understood, in [LN10] it is shown that G is $\mathcal{F}in\,\mathrm{FP}_0$ if and only if G has finitely many conjugacy classes of groups with prime power order and conjectured that a group is $\mathcal{F}in\,\mathrm{FP}_\infty$ if and only if it is FP_∞ and $\mathcal{F}in\,\mathrm{FP}_0$.

As far as we are aware, nothing is known about the conditions $\mathcal{M}_{\mathcal{F}in}\,\mathrm{FP}_n$ or $\mathcal{H}_{\mathcal{F}in}\,\mathrm{FP}_n$.

In Section 2.5 we provide a partial classification of a new condition related to $\mathcal{O}_{\mathcal{F}}\,\mathrm{FP}_n$, called covariant $\mathcal{O}_{\mathcal{F}}\,\mathrm{FP}_n$ and in Section 3.5 we make some observations about the conditions $\mathcal{M}_{\mathcal{F}in}\,\mathrm{FP}_n$, completely describing the condition $\mathcal{M}_{\mathcal{F}in}\,\mathrm{FP}_0$. In Section 4.3, we prove that $\mathcal{H}_{\mathcal{F}in}\,\mathrm{FP}_n$ implies $\mathcal{F}in\,\mathrm{FP}_n$.

0.3 Bredon Duality Groups

A *duality group* is a group G of type FP for which

$$H^i(G, \mathbb{Z}G) \cong \begin{cases} \mathbb{Z}\text{-flat} & \text{if } i = n \\ 0 & \text{else.} \end{cases}$$

Where n is necessarily the cohomological dimension of G. The name duality comes from the fact that this condition is equivalent to existence of a $\mathbb{Z}G$ module D, giving an isomorphism

$$H^i(G, M) \cong H_{n-i}(G, D \otimes_{\mathbb{Z}} M) \qquad (*)$$

for all i and all $\mathbb{Z}G$-modules M. It can be proven that given such an isomorphism, the module D is necessarily $H^n(G, \mathbb{Z}G)$. Such groups were first considered by Bieri and Eckmann [BE73], see also [Bro94, VII.10][Dav00][Bie81, III] for an introduction to these groups. A duality group G is called a *Poincaré duality* group if in addition

$$H^i(G, \mathbb{Z}G) \cong \begin{cases} \mathbb{Z} & \text{if } i = n \\ 0 & \text{else.} \end{cases}$$

These groups are so called because having a manifold model for BG implies G is a Poincaré duality group. Wall posed the question of whether the converse is true [Wal79] - the answer is no as Poincaré duality groups can be built which are not finitely presented [Dav98, Theorem C] - but the question remains a significant open problem if we ask include the requirement that G be finitely presented.

These notions can be generalised to duality groups over R, where R is some commutative ring, so that G is duality over R if G is FP over R and

$$H^i(G, RG) \cong \begin{cases} R\text{-flat} & \text{if } i = n \\ 0 & \text{else.} \end{cases}$$

and G is Poincaré duality over R if

$$H^i(G, RG) \cong \begin{cases} R & \text{if } i = n \\ 0 & \text{else.} \end{cases}$$

The correct analog of Wall's conjecture is whether every torsion-free finitely presented Poincaré duality group over R the fundamental group of an aspherical closed R-homology manifold [Dav00, Question 3.5]. This is answered by Fowler, who describes a Poincaré duality group over $\mathbb{Q}$ which is not the fundamental group of an aspherical closed $\mathbb{Q}$-homology manifold [Fow12]. There are also various sufficient conditions for the conjecture to be true, see the discussion in [Dav00].

Duality groups behave very well under various group theoretic constructions. In particular they are preserved under extension and certain free products with amalgamation [Bie81, III].

If G is a group which admits a cocompact manifold model M for $\mathrm{E}_{\mathcal{F}in}G$, such that for any finite subgroup H, M^H is a submanifold, we have the following condition on the cohomology of the Weyl groups WH [Bro94, VIII.8.2]

$$H^i(WH, \mathbb{Z}[WH]) = \begin{cases} \mathbb{Z} & \text{if } i = \dim M^H \\ 0 & \text{else.} \end{cases}$$

Building on this, in [DL03][MP13, Definition 5.1] a Bredon duality group is defined as an $\mathcal{O}_{\mathcal{F}in}$ FP group G such that for every finite subgroup H of G there is an integer n_H such that

$$H^i(WH, R[WH]) = \begin{cases} R\text{-flat} & \text{if } i = n_H \\ 0 & \text{else.} \end{cases}$$

Furthermore, G is said to be Bredon-Poincaré-duality over R if for all finite H,

$$H^{n_H}(WH, R[WH]) = R$$

Note that for torsion-free groups this reduces to the usual definition of duality and Poincaré-duality groups. Interestingly, there appears to be no analog of the duality described by $(*)$ between homology and cohomology groups seen in the case of ordinary duality.

In Section 5, we give various examples of duality groups, classify some duality groups of low dimension and discuss under what conditions the property of being Bredon duality is preserved by extensions and amalgamated free products.

1 Modules over a category

Much of this section is based on [Lüc89]. Although we consider a slightly more general situation, as explained in Remark 1.1, the idea is the same.

Let R be a commutative ring with unit and $\mathfrak{C}$ a small **Ab** category (sometimes called a preadditive category) with the condition below.

(A) For any two objects x and y in $\mathfrak{C}$, the set of morphisms, denoted $[x, y]_{\mathfrak{C}}$, between x and y is a free Abelian group.

Recall that an **Ab** category is one where the morphisms between any two objects form an abelian group and where morphism composition distributes over this addition: For $w, x, y, z \in \mathfrak{C}$ and morphisms

$$w \xrightarrow{f} x \underset{g'}{\overset{g}{\rightrightarrows}} y \xrightarrow{h} z$$

there is a distributive law

$$h(g + g')f = hgf + hg'f$$

To make $\mathfrak{C}$ an additive category we would require a zero object and binary biproducts, in general though the categories we want to work with won't have this extra structure. The interested reader should consult [Wei94, A.4] for a discussion of these conditions.

Remark 1.1. In [Lüc89, 9.2], categories $\mathfrak{X}$ are considered with the property that every endomorphism in $\mathfrak{X}$ is an isomorphism, then in constructions where we would use the set $[x, y]_{\mathfrak{C}}$, Lück instead uses the free abelian group with basis the morphisms between x and y in $\mathfrak{X}$ (see for example, [Lüc89, 9.8]). Thus the correct analog of Lück's property with our definitions is the following:

(EI) For every $x \in \mathfrak{C}$, the basis elements of $[x, x]_{\mathfrak{C}}$ are isomorphisms.

The main advantage of the (EI) property is that it allows objects in $\mathfrak{C}$ to be given a partial order: setting $x \leq y$ if $[x, y]_{\mathfrak{C}}$ is non-empty. We choose not to ask for this property in this section, since we want everything discussed here to be relevant to the Mackey category, discussed in Section 3, which does not have (EI). The main example of a category with (EI) is the Orbit category, see Example 1.10 and more generally Section 2.

For examples of categories which have (A) but don't have (EI), see Sections 3 and 4, where the Mackey and Hecke categories are considered, neither of which have (EI).

Throughout, the letters $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$ etc. will always denote small **Ab** categories with (A).

Define the category of covariant $\mathfrak{C}$-modules over R to be the category of R-additive covariant functors $\mathfrak{C} \to$ **R-Mod**, the category of left R-modules. Similarly the category of contravariant $\mathfrak{C}$-modules over R is the category of R-additive contravariant functors $\mathfrak{C} \to$ **R-Mod**.

Remark 1.2. If neither "covariant" or "contravariant" is specified in a statement about $\mathfrak{C}$-modules, the statement holds for both covariant and contravariant modules.

Immediately from the definition we have two important facts. Firstly, because $\mathfrak{C}$-modules form a functor category and **R-Mod** is an Abelian category, the category of $\mathfrak{C}$-modules is an Abelian category [Mur06, 44]. In fact, it inherits all of Grothendieck's axioms for an Abelian category which are satisfied by **R-Mod** [Mur06, 44,55], namely:

1. AB3 and AB4 - Every small colimit exists and products of exact sequences are exact.
2. AB3* and AB4* - Every small limit exists and coproducts of exact sequences are exact.
3. AB5 Filtered colimits of exact sequences are exact.

See [Wei94] for a discussion of these axioms. Secondly, again because we are working in a functor category:

Remark 1.3. A sequence of $\mathfrak{C}$-modules

$$0 \longrightarrow A(-) \longrightarrow B(-) \longrightarrow C(-) \longrightarrow 0$$

is exact if and only if it is exact when evaluated at every $x \in \mathfrak{C}$. Note that 0 denotes the zero functor, sending every object to the zero module. Similarly, using the fact that the category of $\mathfrak{C}$-modules is a functor category and the category of Abelian groups is complete, limits and colimits are computed pointwise [Mur06, p.8].

Since $[x, y]_{\mathfrak{C}}$ is Abelian for all x and y in $\mathfrak{C}$, for any $y \in \mathfrak{C}$ we can form a contravariant module $R[-, y]_{\mathfrak{C}}$ by

$$R[-, y]_{\mathfrak{C}}(x) = R \otimes_{\mathbb{Z}} [x, y]_{\mathfrak{C}}$$

The analogous construction for covariant modules gives us a module $R[y, -]_{\mathfrak{C}}$

$$R[y, -]_{\mathfrak{C}}(x) = R \otimes_{\mathbb{Z}} [y, x]_{\mathfrak{C}}$$

Later on in Section 1.2 we will show that these modules are the correct analog of free modules in the category of $\mathfrak{C}$-modules. Since $R[x, y]_{\mathfrak{C}}$ is a free R-module we will usually write $r\alpha$ instead of $r \otimes \alpha$, for $r \in R$ and $\alpha \in [x, y]_{\mathfrak{C}}$.

Remark 1.4. If $f \in R[x, y]_{\mathfrak{C}}$, where $f = \sum_i r_i f_i$ for some $f_i \in [x, y]_{\mathfrak{C}}$, and $Q(-)$ is a $\mathfrak{C}$-module, then we will write $Q(f)$ for the sum:

$$Q(f) = \sum_i r_i Q(f_i)$$

Notice we now have the equality $M(rf) = rM(f)$ for $f \in R[x, y]_{\mathfrak{C}}$ and $r \in R$.

Lemma 1.5 The Yoneda-type Lemma
For any covariant functor $A(-)$ and $x \in \mathfrak{C}$, there is an isomorphism, natural in $A(-)$:

$$\begin{aligned} \mathrm{Mor}_{\mathfrak{C}}\left(R[x,-]_{\mathfrak{C}}, A(-)\right) &\cong A(x) \\ f &\mapsto f(x)(\mathrm{id}_x) \end{aligned}$$

Similarly for any contravariant functor $M(-)$ and $x \in \mathfrak{C}$, there is an isomorphism, natural in $M(-)$:

$$\begin{aligned} \mathrm{Mor}_{\mathfrak{C}}\left(R[-,x]_{\mathfrak{C}}, M(-)\right) &\cong M(x) \\ f &\mapsto f(x)(\mathrm{id}_x) \end{aligned}$$

The proof is a direct translation of the standard proof for the orbit category into the setting of $\mathfrak{C}$-modules, see for example [MV03, p.9].

Proof. We provide a proof only for covariant modules, that for contravariant modules is similar.

Let f be a morphism $f : R[x,-]_{\mathfrak{C}} \to A(-)$, f is completely determined by $f(x)$: If $\alpha \in R[x,y]_{\mathfrak{C}}$ then we can view α as

$$\alpha = \sum_i r_i \alpha_i = \sum_i r_i R[x, \alpha_i](\mathrm{id}_x)$$

Where $\alpha_i[x,y]_{\mathfrak{C}}$ and $r_i \in R$. Thus

$$\begin{aligned} f(y)(\alpha) &= f(y)\left(\sum_i r_i R[x,\alpha_i](\mathrm{id}_x)\right) \\ &= \sum_i r_i f(y) \circ R[x,\alpha_i](\mathrm{id}_x) \\ &\cong \sum_i r_i A(\alpha_i) \circ f(x)(\mathrm{id}_x) \\ &= A(\alpha) \circ f(x)(\mathrm{id}_x) \end{aligned}$$

Where the second equality is because f is R-additive, and the isomorphism is because f is a morphism in the category of $\mathfrak{C}$-modules - so a natural transformation of functors - meaning the diagram below commutes.

$$\begin{array}{ccc} R[x,x]_{\mathfrak{C}} & \xrightarrow{f(x)} & A(x) \\ \Big\downarrow{\scriptstyle R[x,\alpha]_{\mathfrak{C}}} & & \Big\downarrow{\scriptstyle A(\alpha)} \\ R[x,y]_{\mathfrak{C}} & \xrightarrow{f(y)} & A(y) \end{array}$$

Conversely, given an element $a \in A(x)$ we can define a morphism f, with $f(x)(\mathrm{id}_x) = a$, by

$$f(y)(\alpha) = A(\alpha)(a)$$

for any $\alpha \in [x,y]_{\mathfrak{C}}$. □

The endomorphisms $[x,x]_{\mathfrak{C}}$ of an object $x \in \mathfrak{C}$ form an associative ring. This ring will appear often, so much so that we write $\mathrm{End}(x)$ instead of $[x,x]_{\mathfrak{C}}$, and write $R\,\mathrm{End}(x)$ for $R \otimes_{\mathbb{Z}} \mathrm{End}(x)$. We can also consider the ring of automorphisms of an object x in $\mathfrak{C}$, denoted $\mathrm{Aut}(x)$ and the group ring $R\,\mathrm{Aut}(x)$. The (EI) property of Remark 1.1 can now be restated as "$\mathrm{End}(x) = \mathrm{Aut}(x)$ for all $x \in \mathfrak{C}$".

Remark 1.6. Given a covariant module $A(-)$, evaluating $A(-)$ at x gives a left $R\,\mathrm{End}(x)$-module, using the action below (recall the notation described in Remark 1.4):

$$\begin{aligned} R\,\mathrm{End}(x) \times A(x) &\longrightarrow A(x) \\ (f,a) &\longmapsto A(f)(a) \end{aligned}$$

This is a left-module structure since given two elements $g, f \in R\,\mathrm{End}(x)$,

$$(g \circ f) \cdot x = A(g \circ f)(x) = A(g) \circ A(f)(x) = g \cdot (f \cdot x)$$

Similarly, for a contravariant module $M(-)$, $M(x)$ has a right $R\,\mathrm{End}(x)$-module structure.

Remark 1.7. $\mathrm{End}(x)$ could be viewed as a category with one object, and with morphisms the free Abelian group $\mathrm{End}(x)$, denote this category $\widehat{\mathrm{End}(x)}$ to distinguish it from $\mathrm{End}(x)$. Clearly $\widehat{\mathrm{End}(x)}$ has property (A). It's now possible to pass freely between covariant $\widehat{\mathrm{End}(x)}$-module and left $R\,\mathrm{End}(x)$-modules, similarly between contravariant $\widehat{\mathrm{End}(x)}$-modules and right $R\,\mathrm{End}(x)$-modules. Exactly the same statement holds replacing endomorphisms with automorphisms.

There is often a need to consider bi-modules. A $\mathfrak{C}$-$\mathfrak{D}$ bi-module (can be covariant or contravariant in either variable, although most of the bi-modules we shall use will be covariant in one variable in contravariant in the other), is simply a functor

$$Q(-,?) : \mathfrak{C} \times \mathfrak{D} \to \textbf{R-Mod}$$

Example 1.8. Perhaps the most common bi-module we'll come across is the $\mathfrak{C}$-$\mathfrak{C}$ bi-module $R[-,?]_{\mathfrak{C}}$, contravariant in the first variable and covariant in the second.

$$R[-,?]_{\mathfrak{C}} : (x,y) \mapsto R[x,y]_{\mathfrak{C}}$$

Using Remark 1.6 and Example 1.8, $R[x,?]_{\mathfrak{C}}$ is a left $R\,\mathrm{End}(x)$ module, and $R[-,x]_{\mathfrak{C}}$ is a right $R\,\mathrm{End}(x)$-module. Thus we can consider $R[x,?]_{\mathfrak{C}}$ as an $R\,\mathrm{End}(x)_{\mathfrak{C}}$-$\mathfrak{C}$ bi-module, and $R[-,x]_{\mathfrak{C}}$ as a $\mathfrak{C}$-$R\,\mathrm{End}(x)$ bi-module.

Remark 1.9. When considering, for example, the morphisms between two $\mathfrak{C}$-$\mathfrak{C}$ bi-modules $Q(-,?)$ and $P(\dagger,??)$, it can be unclear with respect to which variables we are working with - to solve this from now on we'll use the notation

$$\mathrm{Mor}_{\mathfrak{C}}(Q(\not-,?), P(\not-,??))$$

to indicate that the morphisms are considered with respect to the first variable in each bi-module. For example, in this new notation, the natural isomorphism of the Yoneda-type Lemma 1.5 becomes

$$\mathrm{Mor}_{\mathfrak{C}}\left(R[\not-,x]_{\mathfrak{C}}, M(\not-)\right) \cong M(x)$$

Example 1.10. The Orbit Category $\mathcal{O}_\mathcal{F}$.

The Orbit Category $\mathcal{O}_\mathcal{F}$ is the prototypical example of a category with property (A), and will be studied properly in Section 2. It was introduced for finite groups by Bredon [Bre67], who used the associated cohomology theory, Bredon cohomology, to develop equivariant obstruction theories. It was later generalised to arbitrary groups by Lück [Lüc89].

Fix a family $\mathcal{F}$ of subgroups of G, closed under subgroups and conjugation. Commonly studied families are those of all finite subgroups, and of all virtually cyclic subgroups. The objects of the orbit category $\mathcal{O}_\mathcal{F}$ are all transitive G-sets with stabilisers in $\mathcal{F}$, ie. the G-sets G/H where H is a subgroup in $\mathcal{F}$. The morphism set $[G/H, G/K]_{\mathcal{O}_\mathcal{F}}$ is the free abelian group on the set of G-maps $G/H \to G/K$. A G-map

$$\begin{aligned} \alpha : G/H &\longrightarrow G/K \\ H &\longmapsto gK \end{aligned}$$

is completely determined by the element $\alpha(H) = gK$, and such an element $gK \in G/K$ determines a G-map if and only if $HgK = gK$, usually written as $gK \in (G/K)^H$. Equivalently gK determines a G-map if and only if $g^{-1}Hg \leq K$. Notice that the isomorphism classes of elements in $\mathcal{O}_\mathcal{F}$, denoted $\operatorname{Iso}\mathcal{O}_\mathcal{F}$, are exactly the conjugacy classes of subgroups in $\mathcal{F}$. The orbit category can be thought of as encoding the finite subgroup structure of G.

The orbit category as described above is often written $\mathbb{Z}\,\mathcal{O}_\mathcal{F}$ instead of simply $\mathcal{O}_\mathcal{F}$. Some authors use $\mathcal{O}_\mathcal{F}$ to denote the category with the same objects and whose morphism set is all G-maps (without taking the free abelian group on them) and use $\mathbb{Z}\,\mathcal{O}_\mathcal{F}$ to denote the category we have described. We've chosen this notation so it matches that used for the Mackey and Hecke categories used later.

1.1 Tensor Products

This section describes various tensor products of $\mathfrak{C}$-modules, they can be thought of as generalisations of the tensor product over group rings. Given a group ring RG, for R a commutative ring, a left RG-module A and a right RG-module M the tensor product over RG is the R-module $M \otimes_{RG} A$. The analog for a contravariant $\mathfrak{C}$-module $M(-)$ and covariant $\mathfrak{C}$-module $A(-)$ will be an R-module $M(\not-) \otimes_\mathfrak{C} A(\not-)$, called the *tensor product over* $\mathfrak{C}$. This is described in Section 1.1.1

Given two left RG-modules M and N, the tensor product $M \otimes_R N$ can be given the diagonal action of RG, the $\mathfrak{C}$-module analog of this is the *tensor product over* R. This is the $\mathfrak{C}$-module denoted $A(-) \otimes_R B(-)$, where $A(-)$ and $B(-)$ are either both contravariant or both covariant modules, and is described in Section 1.1.2.

1.1.1 Tensor Product over $\mathfrak{C}$

We describe a construction, due to Lück [Lüc89, 9.12], of the categorical tensor product of [Sch70, 16.7][Fis68] for the categories of $\mathfrak{C}$-modules over R.

The categorical tensor product is the R-module $M(\not-) \otimes_\mathfrak{C} A(\not-)$ such that $M(\not-) \otimes_\mathfrak{C} \dagger$ is left adjoint to $\operatorname{Mor}_\mathfrak{C}(M(\not-), \dagger)$. This definition is valid in functor categories under some technical conditions which can be found in the introduction of [Fis68], but in order to keep this readable will be omitted here.

The notation of crossing out of variables is used for the tensor product as with $\mathrm{Mor}_{\mathfrak{C}}$, see Remark 1.9. So for $M(-)$ contravariant and $A(-)$ covariant, the tensor product over $\mathfrak{C}$ of $M(-)$ and $A(-)$ is written

$$\dagger \otimes_{\mathfrak{C}} \dagger\dagger : M(-) \times A(-) \mapsto M(\not{-}) \otimes_{\mathfrak{C}} A(\not{-})$$

There is an adjoint isomorphism, see Proposition 1.13, reminiscent of the adjoint isomorphism for left and right modules over a ring.

$$\mathrm{Mor}_{\mathfrak{D}}(M(\not{?}) \otimes_{\mathfrak{C}} Q(\not{?}, \not{-}), N(\not{-})) \cong \mathrm{Mor}_{\mathfrak{D}}(M(\not{?}), \mathrm{Mor}_{\mathfrak{C}}(Q(\not{?}, \not{-}), N(\not{-})))$$

Here $Q(?, -)$ is an $\mathfrak{D}$-$\mathfrak{C}$-bi-module - a contravariant $\mathfrak{D}$-module in "$-$" and a covariant $\mathfrak{C}$-module in "?".

The construction of the tensor product is as follows:

$$M(\not{-}) \otimes_{\mathfrak{C}} A(\not{-}) = \bigoplus_{x \in \mathfrak{C}} M(x) \otimes_R A(x) \Big/ \sim$$

Where $\alpha^*(m) \otimes a \sim m \otimes \alpha_*(a)$ for all morphisms $\alpha \in [x, y]$ in $\mathfrak{C}$, elements $m \in M(y)$ and $n \in A(x)$, and objects $x, y \in \mathfrak{C}$. The only change passing from $\mathbb{Z}$ to an arbitrary ring R is that the tensor product in the construction is taken over R instead of $\mathbb{Z}$. Since R is commutative, this construction yields an R-module.

Example 1.11. If A is a left $R\,\mathrm{End}(x)$-module and M is a right $R\,\mathrm{End}(x)$-module then, by Remark 1.7, A and M can be regarded as covariant and contravariant $\widehat{\mathrm{End}(x)}$-modules $A(-)$ and $M(-)$. It's easy to check that

$$M(\not{-}) \otimes_{\widehat{\mathrm{End}(x)}} A(\not{-}) \cong M \otimes_{R\,\mathrm{End}(x)} A$$

Lemma 1.12 [MV03, p.14] There are natural isomorphisms for any contravariant module $M(-)$ and covariant module $A(-)$:

$$M(\not{-}) \otimes_{\mathfrak{C}} R[x, \not{-}]_{\mathfrak{C}} \cong M(x)$$

$$R[\not{-}, x]_{\mathfrak{C}} \otimes_{\mathfrak{C}} A(\not{-}) \cong A(x)$$

Proposition 1.13 [Lüc89, p.166][MP02] There are adjoint natural isomorphisms:

$$\mathrm{Mor}_{\mathfrak{D}}(M(\not{?}) \otimes_{\mathfrak{C}} Q(\not{?}, \not{-}), N(\not{-})) \cong \mathrm{Mor}_{\mathfrak{C}}(M(\not{?}), \mathrm{Mor}_{\mathfrak{D}}(Q(\not{?}, \not{-}), N(\not{-})))$$

$$\mathrm{Mor}_{\mathfrak{C}}(Q(\not{?}, \not{-}) \otimes_{\mathfrak{D}} A(\not{-}), B(\not{?})) \cong \mathrm{Mor}_{\mathfrak{D}}(A(\not{-}), \mathrm{Mor}_{\mathfrak{C}}(Q(\not{?}, \not{-}), B(\not{?})))$$

Here $M(-)$ and $N(-)$ are contravariant modules, $A(-)$ and $B(-)$ are covariant modules, and $Q(?, -)$ is an $\mathfrak{D}$-$\mathfrak{C}$-bi-module - a contravariant $\mathfrak{D}$-module in "$-$" and a covariant $\mathfrak{C}$-module in "?".

Corollary 1.14 There are a natural isomorphisms:

$$\mathrm{Mor}_{\mathfrak{D}}(M \otimes_{R\,\mathrm{End}(x)} R[x, \not-]_{\mathfrak{D}}, N(\not-)) \cong \mathrm{Hom}_{R\,\mathrm{End}(x)}(M, N(x))$$

$$\mathrm{Mor}_{\mathfrak{D}}(R[\not-, x]_{\mathfrak{D}} \otimes_{R\,\mathrm{End}(x)} A, B(\not-)) \cong \mathrm{Hom}_{R\,\mathrm{End}(x)}(A, B(x))$$

Where $M(-)$ and $N(-)$ are arbitrary contravariant modules, and $A(-)$ and $B(-)$ arbitrary covariant modules.

Proof. Specialise Proposition 1.13 to the case $\mathfrak{C} = \widehat{\mathrm{End}(x)}$ and $Q(?, -) = R[x, -]$, recalling from Remark 1.6 that an $R\,\mathrm{End}(x)$-module is equivalently an $\widehat{\mathrm{End}(x)}$-module, where $\widehat{\mathrm{End}(x)}$ is $\mathrm{End}(x)$ viewed as a category with a single element. Thus

$$\begin{aligned} &\mathrm{Mor}_{\mathfrak{D}}\left(M \otimes_{R\,\mathrm{End}(x)} R[x, \not-]_{\mathfrak{D}}, N(\not-)\right) \\ &\qquad \cong \mathrm{Hom}_{R\,\mathrm{End}(x)}\left(M, \mathrm{Mor}_{\mathfrak{D}}\left(R[x, \not-]_{\mathfrak{D}}, N(\not-)\right)\right) \\ &\qquad \cong \mathrm{Hom}_{R\,\mathrm{End}(x)}\left(M, N(x)\right) \end{aligned}$$

Where the second natural isomorphism is the Yoneda-type Lemma 1.5. The other natural isomorphism is proved analogously. □

Lemma 1.15 [MP02, Lemma 3.1] For any contravariant $\mathfrak{D}$-module $M(-)$, $\mathfrak{D}$-$\mathfrak{C}$ bi-module $Q(-, ?)$ (covariant in "-" and contravariant in "?") and covariant module $A(?)$, there is a natural isomorphism:

$$\left(M(\not-) \otimes_{\mathfrak{D}} Q(\not-, \not?)\right) \otimes_{\mathfrak{C}} A(\not?) \cong M(\not-) \otimes_{\mathfrak{D}} \left(Q(\not-, \not?) \otimes_{\mathfrak{C}} A(\not?)\right)$$

Lemma 1.16 The tensor product over $\mathfrak{C}$ commutes with arbitrary direct sums.

Proof. This is clear from the construction. □

Remark 1.17. Occasionally we will be in a situation like the above, except that $Q(-.?)$ is a $R\,\mathrm{End}(x)$-$\mathfrak{C}$ bi-module or similar. For example, $Q = R[x, ?]_{\mathfrak{C}}$. In this case the associativity of the Lemma above becomes, for a covariant $\mathfrak{C}$-module $A(-)$, $R\,\mathrm{End}(x)$-$\mathfrak{C}$ bi-module $Q(-)$ (contravariant in "-"), and right $R\,\mathrm{End}(x)$-module N:

$$\left(N \otimes_{\mathrm{End}(x)} Q(\not-)\right) \otimes_{\mathfrak{C}} A(\not-) \cong N \otimes_{\mathrm{End}(x)} \left(M(\not-) \otimes_{\mathfrak{C}} A(\not-)\right)$$

Similarly for a contravariant $\mathfrak{C}$-module $M(-)$, $R\,\mathrm{End}(x)$-$\mathfrak{C}$ bi-module $Q(-)$ (covariant "-"), and left $R\,\mathrm{End}(x)$-module A:

$$M(\not-) \otimes_{\mathfrak{C}} \left(Q(\not-) \otimes_{\mathrm{End}(x)} A\right) \cong \left(M(\not-) \otimes_{\mathfrak{C}} Q(\not-)\right) \otimes_{\mathrm{End}(x)} A$$

1.1.2 Tensor Product over R

We describe the tensor product over R as in [Lüc89, 9.13]. If $A(-)$ and $B(-)$ are $\mathfrak{C}$-modules, either both covariant or both contravariant, then the tensor product over R is written

$$\dagger \otimes_R \dagger : A(-) \times B(-) \mapsto A(-) \otimes_R B(-)$$

Where

$$A(-) \times_R B(-) : x \longmapsto A(x) \otimes_R B(x)$$

and if $\alpha : x \to y$ is a morphism in $\mathfrak{C}$, then

$$A(-) \otimes_R B(-) : \alpha \longmapsto A(\alpha) \otimes_R B(\alpha)$$

1.2 Frees, Projectives, Injectives and Flats

Free objects in some category are usually defined as left adjoint to some functor, often with codomain **Set**. For modules over a category $\mathfrak{C}$ the necessary forgetful functor is

$$\begin{aligned} U : \{\ \mathfrak{C}\text{-modules}\ \} &\longrightarrow [\mathrm{Ob}(\mathfrak{C}), \mathbf{Set}] \\ UA : x &\longmapsto A(x) \end{aligned}$$

Here $[\mathrm{Ob}(\mathfrak{C}), \mathbf{Set}]$ denotes the category of functors $\mathrm{Ob}(\mathfrak{C}) \to \mathbf{Set}$, where $\mathrm{Ob}(\mathfrak{C})$ is the category whose objects are the objects of $\mathfrak{C}$ but with only the identity morphisms at each object. The functor F left adjoint to U is fairly easy to describe: If $X \in [\mathrm{Ob}(\mathfrak{C}), \mathbf{Set}]$ then

$$FX = \bigoplus_{x \in \mathfrak{C}} \bigoplus_{X(x)} R[x, -]_{\mathfrak{C}}$$

Analagously if we are working with contravariant functors,

$$FX = \bigoplus_{x \in \mathfrak{C}} \bigoplus_{X(x)} R[-, x]_{\mathfrak{C}}$$

That (F, U) form an adjoint pair is a consequence of the Yoneda-type Lemma 1.5. For any covariant module $A(-)$:

$$\begin{aligned} \mathrm{Mor}_{\mathfrak{C}}(FX(-), A(-)) &= \mathrm{Mor}_{\mathfrak{C}}\left(\bigoplus_{x \in \mathfrak{C}} \bigoplus_{X(x)} R\,[x, -]_{\mathfrak{C}}\,, A(-) \right) \\ &\cong \prod_{x \in \mathfrak{C}} \prod_{X(x)} \mathrm{Mor}_{\mathfrak{C}}\left(R\,[x, -]_{\mathfrak{C}}\,, A(-) \right) \\ &\cong \prod_{x \in \mathfrak{C}} \prod_{X(x)} A(x) \\ &\cong \mathrm{Hom}_{[\mathrm{Ob}(\mathfrak{C}), \mathbf{Set}]}(X, UA) \end{aligned}$$

The proof for contravariant functors is analogous.

Projective and injective modules are defined as in any Abelian category, see for instance [Wei94, §2.2]. Free modules are projective: If

$$0 \longrightarrow A(-) \longrightarrow B(-) \longrightarrow C(-) \longrightarrow 0$$

is an exact sequence of $\mathfrak{C}$-modules then, by the Yoneda-type Lemma (1.5), applying $\mathrm{Mor}_{\mathfrak{C}}(R[x, \not?]_{\mathfrak{C}}, -)$ gives the exact sequence

$$0 \longrightarrow A(x) \longrightarrow B(x) \longrightarrow C(x) \longrightarrow 0$$

Since direct sums of projectives are projective in any Abelian category, this is enough to show the category of $\mathfrak{C}$-modules has enough projectives, in fact the counit of the adjunction between F and U

$$\eta : (FUA)(-) \longrightarrow A(-)$$

is always an epimorphism: By construction,

$$FUA(-) = \bigoplus_{x \in \mathfrak{C}} \bigoplus_{a \in A(x)} F_a(x, -)$$

where $F_a(x,-) \cong R[x,-]_{\mathfrak{C}}$. The counit is the map defined on $F_a(x,-)$, via the Yoneda-type Lemma 1.5, by $\mathrm{id}_x \to a$. It's clear that every $a \in A(x)$ is in the image of $\eta(x)$, and thus η is an epimorphism.

The category of $\mathfrak{C}$-module also has enough injectives, see Remark 1.22 for a proof using Coinduction.

A covariant (respectively contravariant) $\mathfrak{C}$-module $F(-)$ is flat if the functor $\dagger \otimes_{\mathfrak{C}} F(\not-)$ (repsectively $F(\not-) \otimes_{\mathfrak{C}} \dagger$) is flat. Lemma 1.12 shows free modules are flat, and since the tensor product commutes with direct sums (Lemma 1.16), projectives are flat also.

1.3 Restriction, Induction and Coinduction

In [Lüc89, §9.8] Lück defines functors called "extension" and "restriction" for any element $x \in \mathfrak{C}$, taking an $R\,\mathrm{End}(x)$-module to a $\mathfrak{C}$-module and vice versa. We define three functors, called restriction, induction and coinduction. Given a functor $\iota : \mathfrak{C} \to \mathfrak{D}$, restriction takes $\mathfrak{D}$-modules to $\mathfrak{C}$-modules and induction and coinduction take $\mathfrak{C}$-modules to $\mathfrak{D}$-modules. In the case that $\iota : \widehat{\mathrm{End}(x)} \hookrightarrow \mathfrak{C}$ is the obvious full functor, the induction and restriction functors agree with Lück's extension and restriction functors. Our naming of these functors follows [MPN06], where induction, restriction and coinduction are defined in this way using functors ι. In almost all cases we consider ι will be a full functor "including" one category in another. Perhaps the main feature of these functors is that induction is left adjoint to restriction and coinduction is right adjoint to restriction.

Remark 1.18. In [Lüc89, §9.8], Lück also defines an adjoint pair of functors called "splitting" and "inclusion". We don't define these here as the adjointness of these functors relies on the (EI) property which we are not assuming holds in our category $\mathfrak{C}$, see Remark 1.1.

Restriction, induction, and coinduction are, for covariant functors:

$$\mathrm{Res}_\iota : \{\text{Covariant } \mathfrak{D}\text{-modules}\} \longrightarrow \{\text{Covariant } \mathfrak{C}\text{-modules}\}$$
$$\mathrm{Res}_\iota : A(-) \longmapsto A \circ \iota(-)$$

$$\mathrm{Ind}_\iota : \{\text{Covariant } \mathfrak{C}\text{-modules}\} \longrightarrow \{\text{Covariant } \mathfrak{D}\text{-modules}\}$$
$$\mathrm{Ind}_\iota : A(-) \longmapsto R[\iota(\not?), -]_{\mathfrak{D}} \otimes_{\mathfrak{C}} A(\not?)$$

Where the notation $R[\iota(?),-]_{\mathfrak{D}}$ means that in the variable "?", this functor should be regarded as a $\mathfrak{C}$-module using ι. Finally, coinduction:

$$\begin{aligned}\mathrm{CoInd}_\iota : \{\text{Covariant } \mathfrak{C}\text{-modules}\} &\longrightarrow \{\text{Covariant } \mathfrak{D}\text{-modules}\} \\ \mathrm{CoInd}_\iota : A(-) &\longmapsto \mathrm{Mor}_{\mathfrak{C}}(R[-,\iota(\not?)]_{\mathfrak{D}}, A(\not?))\end{aligned}$$

For contravariant functors, the definition of restriction is identical, and for induction and coinduction is nearly identical:

$$\begin{aligned}\mathrm{Ind}_\iota : \{\text{Contravariant } \mathfrak{C}\text{-modules}\} &\longrightarrow \{\text{Contravariant } \mathfrak{D}\text{-modules}\} \\ \mathrm{Ind}_\iota : M(-) &\longmapsto M(\not?) \otimes_{\mathfrak{C}} R[-,\iota(\not?)]_{\mathfrak{D}}\end{aligned}$$

$$\begin{aligned}\mathrm{CoInd}_\iota : \{\text{Contravariant } \mathfrak{C}\text{-modules}\} &\longrightarrow \{\text{Contravariant } \mathfrak{D}\text{-modules}\} \\ \mathrm{CoInd}_\iota : M(-) &\longmapsto \mathrm{Mor}_{\mathfrak{C}}(R[\iota(\not?),-]_{\mathfrak{D}}, M(\not?))\end{aligned}$$

Usually the functor ι will be implicit, and we will use the notation $\mathrm{Res}^{\mathfrak{D}}_{\mathfrak{C}}$ for Res_ι, and similarly for induction and coinduction. We will also write $\mathrm{Res}^{\mathfrak{C}}_x$ instead of $\mathrm{Res}^{\mathfrak{C}}_{\widehat{\mathrm{End}(x)}}$ and similarly for induction and coinduction.

A basic but very useful fact about induction and coinduction is that for any left $R\,\mathrm{End}(x)$-module A,

$$\mathrm{Ind}^{\mathfrak{C}}_x A(x) = R[x,x] \otimes_{R\,\mathrm{End}(x)} A \cong A$$

$$\mathrm{CoInd}^{\mathfrak{C}}_x A(x) = \mathrm{Hom}_{R\,\mathrm{End}(x)}(R[x,x], A) \cong A$$

and similarly for right $R\,\mathrm{End}(x)$-modules and contravariant induction and coinduction.

Another useful observation, and an immediate consequence of Lemma 1.12, is that induction takes frees to frees:

$$\mathrm{Ind}^{\mathfrak{D}}_{\mathfrak{C}} R[\dagger,-]_{\mathfrak{C}} = R[\iota(\not?),-]_{\mathfrak{D}} \otimes_{\mathfrak{C}} R[\dagger,\not?]_{\mathfrak{C}} \cong R[\dagger,-]_{\mathfrak{D}}$$

and similarly for contravariant modules. We'll generalise this fact later in Proposition 1.20, showing that induction preserves projectives.

Using Proposition 1.13 and Lemma 1.5 gives a chain of natural isomorphisms (shown here for covariant modules):

$$\begin{aligned}\mathrm{Mor}_{\mathfrak{D}}\left(\mathrm{Ind}^{\mathfrak{D}}_{\mathfrak{C}} A(-), B(\not-)\right) &\cong \mathrm{Mor}_{\mathfrak{D}}\left(R[\iota(\not?),\not-]_{\mathfrak{D}} \otimes_{\mathfrak{C}} A(\not?), B(-)\right) \\ &\cong \mathrm{Mor}_{\mathfrak{C}}\left(A(\not?), \mathrm{Mor}_{\mathfrak{D}}\left(R[\iota(\not?),\not-]_{\mathfrak{C}}, B(\not-)\right)\right) \\ &\cong \mathrm{Mor}_{\mathfrak{C}}\left(A(\not?), B \circ \iota(\not?)\right) \\ &\cong \mathrm{Mor}_{\mathfrak{C}}\left(A(\not?), \mathrm{Res}^{\mathfrak{D}}_{\mathfrak{C}}(\not?)\right)\end{aligned}$$

Thus induction is left adjoint to restriction. Using Lemma 1.12, restriction can be reformulated as follows

$$\mathrm{Res}^{\mathfrak{D}}_{\mathfrak{C}} A(-) = R[\not?,\iota(-)]_{\mathfrak{D}} \otimes_{\mathfrak{D}} A(\not?)$$

Now, using Proposition 1.13 again gives the adjointness of coinduction and restriction. Here is the proof for covariant modules, that for contravariant is almost exactly the same.

$$\begin{aligned}\mathrm{Mor}_{\mathfrak{C}}\left(\mathrm{Res}_{\mathfrak{C}}^{\mathfrak{D}} A(\not{-}), B(\not{-})\right) &\cong \mathrm{Mor}_{\mathfrak{C}}\left(R[\not{?}, \iota(\not{-})]_{\mathfrak{D}} \otimes_{\mathfrak{D}} A(\not{?}), B(\not{-})\right) \\ &\cong \mathrm{Mor}_{\mathfrak{D}}\left(A(\not{?}), \mathrm{Mor}_{\mathfrak{C}}\left(R[\not{?}, \iota(\not{-})]_{\mathfrak{D}}, B(\not{-})\right)\right) \\ &\cong \mathrm{Mor}_{\mathfrak{D}}\left(A(\not{?}), \mathrm{CoInd}_{\mathfrak{C}}^{\mathfrak{D}} B(\not{?})\right)\end{aligned}$$

We've shown:

Proposition 1.19 Induction is left adjoint to restriction and coinduction is right adjoint to restriction.

The following proposition is almost entirely a consequence of this adjointness.

Proposition 1.20

1. Restriction is exact.
2. Induction is right exact and preserves projectives, flats and "finitely generated".
3. Coinduction preserves injectives.
4. Induction and restriction preserve colimits and coinduction and restriction preserve limits.

Proof.

1. Since a short exact sequence of modules over $\mathfrak{C}$ is exact if and only if it's exact when evaluated at every element of $\mathfrak{C}$, restriction is always exact.
2. Since induction has an exact right adjoint it preserves projectives [Wei94, 2.3.10] and is right-exact [Wei94, 2.6.1].

   That induction takes flats to flats is a direct consequence of Lemma 1.21 below. In the covariant case, this implies the functor $? \otimes_{\mathfrak{D}} \mathrm{Ind}_{\mathfrak{C}}^{\mathfrak{D}} F(\not{-})$ is naturally isomorphic to the functor $(\mathrm{Res}_{\mathfrak{C}}^{\mathfrak{D}} ?) \otimes_{\mathfrak{C}} F(\not{-})$. Thus if $F(-)$ is assumed flat then $? \otimes_{\mathfrak{D}} \mathrm{Ind}_{\mathfrak{C}}^{\mathfrak{D}} F(-)$ is exact. An analogous proof holds for contravariant $F(-)$.

   If $A(-)$ is a finitely generated $\mathfrak{C}$-module then there is an epimorphism $F(-) \twoheadrightarrow A(-)$ for some finitely generated free $F(-)$. Induction is right exact so there is an epimorphism

   $$\mathrm{Ind}_{\mathfrak{C}}^{\mathfrak{D}} F(-) \twoheadrightarrow \mathrm{Ind}_{\mathfrak{C}}^{\mathfrak{D}} A(-)$$

   Induction takes frees to frees so $\mathrm{Ind}_{\mathfrak{C}}^{\mathfrak{D}} A(-)$ is finitely generated.
3. Since coinduction has an exact left adjoint it preserves injectives [Wei94, 2.3.10] and is left-exact [Wei94, 2.6.1]
4. This is another consequence of adjointness [ML98, p.118].

□

Lemma 1.21 There are natural isomorphisms for any contravariant $\mathfrak{C}$-module $M(-)$ and covariant $\mathfrak{C}$-module $A(-)$.

$$M(\not-)\otimes_{\mathfrak{D}} \operatorname{Ind}_{\mathfrak{C}}^{\mathfrak{D}} A(\not-) \cong \operatorname{Res}_{\mathfrak{C}}^{\mathfrak{D}} M(\not-)\otimes_{\mathfrak{C}} A(\not-)$$

$$\operatorname{Ind}_{\mathfrak{C}}^{\mathfrak{D}} M(\not-)\otimes_{\mathfrak{D}} A(\not-) \cong M(\not-)\otimes_{\mathfrak{C}} \operatorname{Res}_{\mathfrak{C}}^{\mathfrak{D}} A(\not-)$$

Proof. We prove first natural isomorphism, the second is analogous.

$$\begin{aligned} M(\not-)\otimes_{\mathfrak{D}} \operatorname{Ind}_{\mathfrak{C}}^{\mathfrak{D}} A(\not-) &\cong M(\not-)\otimes_{\mathfrak{D}} \big(R[\not?,\not-]_{\mathfrak{D}}\otimes_{\mathfrak{C}} A(\not?)\big) \\ &\cong \big(M(\not-)\otimes_{\mathfrak{D}} R[\not?,\not-]_{\mathfrak{D}}\big)\otimes_{\mathfrak{C}} A(\not?) \\ &\cong \operatorname{Res}_{\mathfrak{C}}^{\mathfrak{D}} M(\not-)\otimes_{\mathfrak{C}} A(\not-) \end{aligned}$$

Where the second natural isomorphism is Lemma 1.15. □

Remark 1.22. In Section 1.2, it was shown that the category of $\mathfrak{C}$-modules has enough projectives, a consequence of Proposition 1.20(3) is that the category of $\mathfrak{C}$-modules has enough injectives as well. For any ring S and module M over S there always exists an injective module I and injection $M \hookrightarrow I$ [Wei94, 2.3.11]. Given a $\mathfrak{C}$-module $M(-)$ choose injective $R\operatorname{End}(x)$-modules I_x such that $M(x)$ injects into I_x for all $x \in \mathfrak{C}$, and consider the map

$$\prod_{x\in\mathfrak{C}} \eta_x : M(-) \longrightarrow \prod_{x\in\mathfrak{C}} \operatorname{CoInd}_{R\operatorname{End}(x)}^{\mathfrak{C}} I_x(-)$$

Where η_x is chosen, via the adjointness of coinduction and restriction, such that $\eta_x(x)$ is the inclusion of $M(x)$ into $\operatorname{CoInd}_x^{\mathfrak{C}} I_x(x) = I_x$. Clearly the product of the η_x maps is an injection. The module on the right is injective by Proposition 1.20(3) and the fact that in any Abelian category, products of injective modules are injective.

Example 1.23. If $A(-)$ and $B(?)$ are covariant $\mathfrak{C}$-modules, we define a $\mathfrak{C}$-$\mathfrak{C}$ bi-module:

$$A(?)\otimes_R B(-) : (x,y) \mapsto A(x)\otimes A(y)$$

Denote by $\Delta : \mathfrak{C} \to \mathfrak{C}\times\mathfrak{C}$ the diagonal functor $\Delta : x \to (x,x)$. The tensor product over R, as defined in Section 1.1.2, could be defined as

$$A(-)\otimes_R B(-) = \operatorname{Res}_{\Delta}(A\otimes_R B)(-)$$

1.4 Tor and Ext

Since the categories of $\mathfrak{C}$-modules are Abelian and have enough projectives, we can do homological algebra with them. If $A(-)$ is a covariant $\mathfrak{C}$-module and $P_*(-)$ a projective resolution of $A(-)$ then for any covariant module $B(-)$ and contravariant module $M(-)$, we define $\operatorname{Ext}_{\mathfrak{C}}^*$ and $\operatorname{Tor}_*^{\mathfrak{C}}$ as one would expect.

$$\operatorname{Ext}_{\mathfrak{C}}^k(A(\not-),B(\not-)) = H^k \operatorname{Mor}_{\mathfrak{C}}\big(P_*(\not-),B(\not-)\big)$$

$$\operatorname{Tor}_k^{\mathfrak{C}}(M(\not-),A(\not-)) = H_k\big(M(\not-)\otimes_{\mathfrak{C}} P_*(\not-)\big)$$

We make the same definitions for contravariant modules: If $M(-)$ is a contravariant module, $Q_*(-)$ a projective resolution of $M(-)$, $A(-)$ a covariant module and $N(-)$ a contravariant module.

$$\mathrm{Ext}^k_{\mathfrak{C}}(M(\not-), N(\not-)) = H^k \,\mathrm{Mor}_{\mathfrak{C}}\big(Q_*(\not-), N(\not-)\big)$$

$$\mathrm{Tor}_k^{\mathfrak{C}}(M(\not-), A(\not-)) = H_k\big(Q_*(\not-) \otimes_{\mathfrak{C}} A(\not-)\big)$$

A priori $\mathrm{Tor}_*^{\mathfrak{C}}$ has just been given two definitions, fortunately there is Proposition 1.25 below, an analog of the classical result that Tor of modules over a ring can be computed using a resolution in either variable.

Remark 1.24. The reason the ring R is not mentioned in the notation for $\mathrm{Tor}_*^{\mathfrak{C}}$ and $\mathrm{Ext}^*_{\mathfrak{C}}$ is that they are essentially independent under change of rings, as explained in Proposition 1.31, if the ring does need to be emphasised the notation used is $\mathrm{Tor}_*^{\mathfrak{C},R}$ and $\mathrm{Ext}^*_{\mathfrak{C},R}$, this will be very rare however.

Proposition 1.25 If $A(-)$ is any covariant module and $M(-)$ is any contravariant module, $P_*(-)$ is a projective covariant resolution of $A(-)$ and $Q_*(-)$ is a projective contravariant resolution of $M(-)$ then for all k,

$$H_k\big(M(\not-) \otimes_{\mathfrak{C}} P_*(\not-)\big) \cong H_k\big(Q_*(\not-) \otimes_{\mathfrak{C}} A(\not-)\big)$$

Showing the two definitons of $\mathrm{Tor}_*^{\mathfrak{C}}$ given are equivalent.

We need some notation for the proof: If $(C_*(-), \partial)$ is an arbitrary chain complex of $\mathfrak{C}$ modules then we write $C_{*+j}(-)$ for the chain complex whose degree i term is $C_{i+j}(-)$, and differential $(-1)^j\partial$ (this is denoted by $C[j]_*(-)$ in [Wei94]). Note that this change in the differential doesn't affect exactness, in fact the homology groups of the new complex are simply $H_n(C_{*+j}(-)) = H_{n+j}(C_*(-))$.

Proof. The proof is a direct translation of [Wei94, Theorem 2.7.2, p.58] into the setting of modules over $\mathfrak{C}$. Form three double complexes, $M(\not-) \otimes_{\mathfrak{C}} P_*(\not-)$, $Q_*(\not-) \otimes_{\mathfrak{C}} P_*(\not-)$ and $Q_*(\not-) \otimes_{\mathfrak{C}} A(\not-)$. The augmentation maps $\varepsilon : P_*(-) \longrightarrow A(-)$ and $\eta : Q_*(-) \longrightarrow M(-)$ induce maps between the total complexes,

$$\mathrm{Tot}\,(Q_*(\not-) \otimes_{\mathfrak{C}} P_*(\not-)) \longrightarrow \mathrm{Tot}\,(M(\not-) \otimes_{\mathfrak{C}} P(\not-)) \cong M(\not-) \otimes_{\mathfrak{C}} P_*(\not-)$$

$$\mathrm{Tot}\,(Q_*(\not-) \otimes_{\mathfrak{C}} P_*(\not-)) \longrightarrow \mathrm{Tot}\,(Q_*(\not-) \otimes_{\mathfrak{C}} A(\not-)) \cong Q_*(\not-) \otimes_{\mathfrak{C}} A_*(\not-)$$

Where Tot denotes the total complex of a bicomplex of R-modules (see [Wei94, 1.2.6] for the definition of total complex). We claim that these maps are weak equivalences. Define a new double complex C_{**}, by adding $A_*(\not-) \otimes_{\mathfrak{C}} Q_{*-1}(\not-)$ in the (-1) column of $P_*(\not-) \otimes_{\mathfrak{C}} Q_*(\not-)$, giving the following complex. Note that we need to shift Q_* so that the resulting complex is a bi-complex, without the shift the horizontal and vertical differentials would not anti-commute.

$$
\begin{array}{ccccccc}
\cdots & & \cdots & & \cdots & & \\
\downarrow & & \downarrow & & \downarrow & & \\
A(\not-) \otimes_{\mathfrak{C}} Q_2(\not-) & \longleftarrow & P_0(\not-) \otimes_{\mathfrak{C}} Q_2(\not-) & \longleftarrow & P_1(\not-) \otimes_{\mathfrak{C}} Q_2(\not-) & \longleftarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
A(\not-) \otimes_{\mathfrak{C}} Q_1(\not-) & \longleftarrow & P_0(\not-) \otimes_{\mathfrak{C}} Q_1(\not-) & \longleftarrow & P_1(\not-) \otimes_{\mathfrak{C}} Q_1(\not-) & \longleftarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
A(\not-) \otimes_{\mathfrak{C}} Q_0(\not-) & \longleftarrow & P_0(\not-) \otimes_{\mathfrak{C}} Q_0(\not-) & \longleftarrow & P_1(\not-) \otimes_{\mathfrak{C}} Q_0(\not-) & \longleftarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}
$$

By inspection, the complex $\mathrm{Tot}(C_{**})_{*+1}$ is the mapping cone of $\varepsilon \otimes_{\mathfrak{C}} \mathrm{id}_Q$, so it suffices to show it is acyclic (see [Wei94, §1.5]). But this follows from the Acyclic Assembly Lemma [Wei94, 2.7.3], since the flatness of $Q_i(-)$ means the functor $\dagger \otimes_{\mathfrak{C}} Q_i(\not-)$ is exact for all i and hence the rows of C_{**} are exact.

Similarly, the mapping cone of $\mathrm{id}_P \otimes_{\mathfrak{C}} \eta$ is the complex $\mathrm{Tot}(D_{**})_{*+1}$, where D_{**} is the double complex obtained by adding $P_{*-1}(\not-) \otimes_{\mathfrak{C}} B(\not-)$ in row (-1) to the complex $P_*(\not-) \otimes_{\mathfrak{C}} Q_*(\not-)$. Since $P_i(-)$ is flat for all i, $P_i(\not-) \otimes_{\mathfrak{C}} \dagger$ is exact, and the columns of $D_{**}(-)$ are exact. Thus $\mathrm{Tot}(D_{**})_{*+1}$ is acyclic, again by the Acyclic Assembly Lemma [Wei94, 2.7.3], showing $\mathrm{id}_P \otimes_{\mathfrak{C}} \eta$ is a weak equivalence.

□

$\mathrm{Tor}_*^{\mathfrak{C}}$ could also be calculated using flat resolutions instead of projective resolutions. The standard proof of this in the case of modules over a ring goes through with almost no modification, see for example [Wei94, 3.2.8]. Similarly, we could calculate $\mathrm{Ext}^*_{\mathfrak{C}}$ using injective resolutions, again the proof is the standard one.

Remark 1.26. We could define a notion of weak dimension of the category of $\mathfrak{C}$-module, mirroring that for modules over a ring, by saying that the weak dimension is the maximal length of a flat resolution of any $\mathfrak{C}$-module, or equivalently

$$\sup\{ i \,:\, \mathrm{Tor}_i^{\mathcal{O}_{\mathcal{F}}}(M(\not-), A(\not-)) \neq 0 \text{ for some modules } A(-) \text{ and } M(-) \,\}$$

Proposition 1.25 then implies the weak dimension of the categories of covariant and contravariant modules coincide, since $\mathrm{Tor}_*^{\mathcal{O}_{\mathcal{F}}}$ can be calculated using covariant or contravariant resolutions.

1.5 Finiteness Conditions

As discussed in Section 1.2, the category of $\mathfrak{C}$-modules has enough free modules, thus for any $\mathfrak{C}$-module $A(-)$ we can build a free resolution of $A(-)$.

$$\cdots \longrightarrow F_1(-) \longrightarrow F_0(-) \longrightarrow A(-) \longrightarrow 0$$

Following ordinary module theory, $A(-)$ is said to be *finitely generated* if $F_0(-)$ can be taken finitely generated, and *finitely presented* if both $F_0(-)$ and $F_1(-)$ can be taken finitely generated.

Lemma 1.27 Every $\mathfrak{C}$-module $A(-)$ is the colimit of its finitely generated submodules.

Proof. Every element $a \in A(x)$ is contained in some finitely generated submodule, namely the image of the map $R[x,-]_{\mathfrak{C}} \to A(-)$ sending $\mathrm{id}_x \mapsto a$. Such a map exists by the Yoneda-type Lemma 1.5. For the contravariant case simply replace $R[x,-]_{\mathfrak{C}}$ with $R[-,x]_{\mathfrak{C}}$. □

We define projective and flat dimension as one would expect, the projective dimension of a contravariant $\mathfrak{C}$-module $A(-)$ is the minimal length of a projective resolution of $A(-)$ and the flat dimension is the minimal length of a flat resolution. These can be characterised as the vanishing of the $\mathrm{Ext}^*_{\mathfrak{C}}$ and $\mathrm{Tor}^{\mathfrak{C}}_*$ groups as is ordinary homological algebra.

We say a $\mathfrak{C}$-module $A(-)$ is $\mathfrak{C}\mathrm{FP}_n$ if there is a projective resolution of $A(-)$ which is finitely generated up to degree n. Clearly $\mathfrak{C}\mathrm{FP}_0$ is the same as finitely generated and $\mathfrak{C}\mathrm{FP}_1$ is the same as finitely presented.

There is an analog of the Bieri-Eckmann criterion of [BE74], see also [Bie81, Theorem 1.3]. A proof in the case that $\mathfrak{C} = \mathcal{O}_{\mathcal{F}}$ appears in [MPN11, Theorem 5.3].

Theorem 1.28 Bieri-Eckmann Criterion
The following conditions on any $\mathfrak{C}$-module $A(-)$ are equivalent:

1. $A(-)$ is $\mathfrak{C}\mathrm{FP}_n$.
2. If $B_\lambda(-)$, for $\lambda \in \Lambda$, is an filtered system of $\mathfrak{C}$-modules then the natural map
$$\varinjlim_{\Lambda} \mathrm{Ext}^k_{\mathfrak{C}}(A(\not-), B_\lambda(\not-)) \longrightarrow \mathrm{Ext}^k_{\mathfrak{C}}(A(\not-), \varinjlim_{\Lambda} B_\lambda(\not-))$$
is an isomorphism for $k \leq n-1$ and a monomorphism for $k = n$.
3. For any filtered system $B_\lambda(-)$, for $\lambda \in \Lambda$, such that $\varinjlim_{\Lambda} B_\lambda(-) = 0$,
$$\varinjlim_{\Lambda} \mathrm{Ext}^k_{\mathfrak{C}}(A(\not-), B_\lambda(\not-)) = 0$$
for all $k \leq n$.

There is also a version of the Bieri-Eckmann criterion using $\mathrm{Tor}^{\mathfrak{C}}_*$ instead of $\mathrm{Ext}^*_{\mathfrak{C}}$, see [Bie81, Theorem 1.3] for the classical case and [MPN11, Theorem 5.4] for the case $\mathfrak{C} = \mathcal{O}_{\mathcal{F}}$.

Proof. 1 ⇒ 2 Choose a free resolution $F_*(-)$ of $A(-)$ by $\mathfrak{C}$-modules, finitely generated up to dimension n and a directed system $B_\lambda(-)$, for $\lambda \in \Lambda$, of $\mathfrak{C}$-modules. Since directed colimits are exact [Wei94, 2.6.15], $\varinjlim$ commutes with the homology functor H^*.
$$\varinjlim_{\Lambda} H^* \mathrm{Mor}_{\mathfrak{C}}(P_*(\not-), B_\lambda(\not-)) \cong H^* \varinjlim_{\Lambda} \mathrm{Mor}_{\mathfrak{C}}(P_*(\not-), B_\lambda(\not-))$$

The result follows from Lemma 1.29, which gives that in the commutative diagram below, the left hand map is an isomorphism and the right hand map an epimorphism.

$$\begin{array}{ccccc} \cdots \to & \varinjlim_\Lambda \mathrm{Mor}_{\mathfrak{C}}(P_{n-1}(-),B_\lambda(-)) & \to & \varinjlim_\Lambda \mathrm{Mor}_{\mathfrak{C}}(P_n(-),B_\lambda(-)) & \to \cdots \\ & \downarrow \cong & & \downarrow & \\ \cdots \to & \mathrm{Mor}_{\mathfrak{C}}(P_{n-1}(-),\varinjlim_\Lambda B_\lambda(-)) & \to & \mathrm{Mor}_{\mathfrak{C}}(P_n(-),\varinjlim_\Lambda B_\lambda(-)) & \to \cdots \end{array}$$

$2 \Rightarrow 3$ This step is obvious.

$2 \Rightarrow 1$ Let $n = 0$ and consider the directed system $A(-)/C_\lambda(-)$, where $C_\lambda(-)$, for $\lambda \in \Lambda$, runs over all finitely generated submodules of $A(-)$. Since any $\mathfrak{C}$-module is the colimit of its finitely generated submodules (Lemma 1.27)

$$\varinjlim_{\Lambda} A(-)/C_\lambda(-) = 0$$

By assumption then

$$\varinjlim_{\Lambda} \mathrm{Mor}_{\mathfrak{C}}(A(-), A(-)/C_\lambda(-)) = 0$$

Thus the canonical projection

$$\pi_\lambda : A(-) \longrightarrow A(-)/C_\lambda(-)$$

is zero in the direct limit, so there exists some $\lambda \in \Lambda$ for which $\pi_\lambda = 0$, thus $C_\lambda(-) = A(-)$ and $A(-)$ is finitely generated.

If $n \geq 1$ then by the above we know $A(-)$ is finitely generated, a dimension shifting argument completes the proof. Pick a finitely generated free $\mathfrak{C}$-module $F(-)$ with an epimorphism onto $A(-)$, giving a short exact sequence:

$$0 \longrightarrow K(-) \longrightarrow F(-) \longrightarrow A(-) \longrightarrow 0$$

Let B_λ be a directed system with $\varinjlim B_\lambda(-) = 0$, by the Ext-long exact sequence,

$$\varinjlim_{\Lambda} \mathrm{Ext}^*_{\mathfrak{C}}(K(-), B_\lambda(-)) = 0$$

for all $k \leq n-1$ and by the induction hypothesis we get that $K(-)$ is $\mathfrak{C}\mathrm{FP}_{n-1}$. Choose a projective resolution $Q_*(-)$ of $K(-)$, finitely generated up to dimension $n-1$, then

$$\cdots \longrightarrow Q_1(-) \longrightarrow Q_0(-) \longrightarrow F(-) \longrightarrow A(-)$$

is the required resolution of $A(-)$, where the map from $Q_0(-)$ to $F(-)$ is the composition

$$Q_0(-) \twoheadrightarrow K(-) \hookrightarrow F(-)$$

□

Lemma 1.29 Given any filtered system $B_\lambda(-)$, for $\lambda \in \Lambda$, of $\mathfrak{C}$-modules, the natural map

$$\varinjlim_{\lambda\in\Lambda} \mathrm{Mor}_{\mathfrak{C}}(A(-), B_\lambda(-)) \longrightarrow \mathrm{Mor}_{\mathfrak{C}}\left(A(-), \varinjlim_{\lambda\in\Lambda} B_\lambda(-)\right)$$

is an epimorphism when $A(-)$ is finitely generated and an isomorphism when $A(-)$ is finitely presented.

Proof. We provide a proof for covariant modules, that for contravariant is similar.

The fact that the natural map is an isomorphism when $A(-)$ is finitely generated free will be needed for the proof. This is because of the following chain of isomorphisms

$$\begin{aligned}
\mathrm{Mor}_{\mathfrak{C}}\left(\bigoplus_{i\in I} R[x_i,-]_{\mathfrak{C}}, \varinjlim_{\lambda\in\Lambda} B_\lambda(-)\right) &\cong \bigoplus_{i\in I} \mathrm{Mor}_{\mathfrak{C}}\left(R[x_i,-]_{\mathfrak{C}}, \varinjlim_{\lambda\in\Lambda} B_\lambda(-)\right) \\
&\cong \bigoplus_{i\in I} \varinjlim_{\lambda\in\Lambda} B(x_i) \\
&\cong \bigoplus_{i\in I} \varinjlim_{\lambda\in\Lambda} \mathrm{Mor}_{\mathfrak{C}}(R[x_i,-]_{\mathfrak{C}}, B_\lambda(-)) \\
&\cong \varinjlim_{\lambda\in\Lambda} \mathrm{Mor}_{\mathfrak{C}}\left(\bigoplus_{i\in I} R[x_i,-]_{\mathfrak{C}}, B_\lambda(-)\right)
\end{aligned}$$

Where the first and last isomorphisms are because $|I| < \infty$ and the second and third are the Yoneda-type Lemma 1.5 and that colimits are computed pointwise.

We can now prove the lemma, choose free $\mathfrak{C}$-modules $F_0(-)$ and $F_1(-)$ with an exact sequence

$$F_1(-) \longrightarrow F_0(-) \longrightarrow A(-) \longrightarrow 0$$

If $A(-)$ is finitely generated we may choose $F_0(-)$ finitely generated and if $A(-)$ is finitely presented then $F_1(-)$ may be chosen finitely generated also. There is a commutative diagram with exact rows

$$\begin{array}{ccccccc}
\varinjlim \mathrm{Mor}_{\mathfrak{C}}(F_1, B_\lambda) & \longrightarrow & \varinjlim \mathrm{Mor}_{\mathfrak{C}}(F_0, B_\lambda) & \longrightarrow & \varinjlim \mathrm{Mor}_{\mathfrak{C}}(A, B_\lambda) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{Mor}_{\mathfrak{C}}(F_1, \varinjlim B_\lambda) & \longrightarrow & \mathrm{Mor}_{\mathfrak{C}}(F_0, \varinjlim B_\lambda) & \longrightarrow & \mathrm{Mor}_{\mathfrak{C}}(A, \varinjlim B_\lambda) & \longrightarrow & 0
\end{array}$$

If $A(-)$ is finitely generated then since $F_0(-)$ is finitely generated, the central vertical map is an isomorphism, and the result follows from the four lemma. If $A(-)$ is finitely presented then $F_0(-)$ and $F_1(-)$ are finitely generated, the central and left hand maps are isomorphisms and the result follows from the five lemma. □

Lemma 1.30 If

$$0 \longrightarrow A(-) \longrightarrow B(-) \longrightarrow C(-) \longrightarrow 0$$

is a short exact sequence of $\mathfrak{C}$-modules then

1. If $A(-)$ and $B(-)$ are $\mathfrak{C}\mathrm{FP}_n$ then $C(-)$ is $\mathfrak{C}\mathrm{FP}_n$.
2. If $A(-)$ and $C(-)$ are $\mathfrak{C}\mathrm{FP}_n$ then $B(-)$ is $\mathfrak{C}\mathrm{FP}_n$.
3. If $B(-)$ and $C(-)$ are $\mathfrak{C}\mathrm{FP}_n$ then $A(-)$ is $\mathfrak{C}\mathrm{FP}_{n-1}$.

Proof. Use the long exact sequence associated to $\mathrm{Ext}^*_{\mathfrak{C}}$ and the Bieri-Eckmann criterion (Theorem 1.28). □

1.6 Change of Rings

If $\varphi : R_1 \to R_2$ is a ring homomorphism then we define the change of rings functor φ^* from $\mathfrak{C}$-modules over R_2 to $\mathfrak{C}$-modules over R_1 as follows.

$$\varphi^* A(-) : x \mapsto A(x)$$

$$\varphi^* A\left(\sum_i r_i \alpha_i\right) = \sum_i \varphi(r_i) A(\alpha_i)$$

Where $r_i \in R_1$ and the α_i are morphisms $x \to y$ for some $x, y \in \mathfrak{C}$.

Such a φ also allows R_2 to be viewed as an R_1-module. If $A(-)$ is a $\mathfrak{C}$-module over R_1 then the tensor product $R_2 \otimes_{R_1} A(-)$ is an $\mathfrak{C}$-module over R_2, where $R_2 \otimes_{R_1} A(-)$ is the $\mathfrak{C}$-module defined by $x \mapsto R_2 \otimes_{R_1} A(x)$. Applying this to a free module

$$R_2 \otimes_{R_1} R_1[x, -]_{\mathfrak{C}} \cong R_2[x, -]_{\mathfrak{C}}$$

Hence if $P(-)$ is a projective $\mathfrak{C}$-module over R_1 then $R_2 \otimes_{R_1} P(-)$ is a projective $\mathfrak{C}$-module over R_2.

Proposition 1.31 If $\varphi : R_1 \to R_2$ is a ring homomorphism and $A(-)$ is a covariant $\mathfrak{C}$-module then

$$\mathrm{Tor}_*^{R_1, \mathfrak{C}}(\underline{R}_1(\not-), \varphi^* A(\not-)) \cong \mathrm{Tor}_*^{R_2, \mathfrak{C}}(\underline{R}_2(\not-), A(\not-))$$

There are similar isomorphisms for contravariant modules and for $\mathrm{Ext}^*_{\mathfrak{C}}$.

Proof. Firstly, consider the case $\varphi : \mathbb{Z} \to R$ for some ring R, we prove

$$\mathrm{Tor}_*^{\mathbb{Z}, \mathfrak{C}}(\underline{\mathbb{Z}}(\not-), \varphi^* A(\not-)) = \mathrm{Tor}_*^{R, \mathfrak{C}}(\underline{R}(\not-), A(\not-))$$

Choose a resolution $P_*(-)$ of $\underline{\mathbb{Z}}(-)$ by contravariant projective $\mathfrak{C}$-modules over $\mathbb{Z}$. For any x in $\mathfrak{C}$, $P_*(x)$ is a $\mathbb{Z}$-split resolution, so applying the functor $R \otimes_{\mathbb{Z}} -$ to $P_*(-)$ yields a projective resolution of $\underline{R}(-)$ by projective $\mathfrak{C}$-modules over R. Observing that

$$P_*(\not-) \otimes_{\mathfrak{C}, \mathbb{Z}} \varphi^* A(\not-) \cong (P_*(\not-) \otimes_{\mathbb{Z}} R) \otimes_{\mathfrak{C}, R} A(\not-)$$

Completes the proof. This isomorphism can be seen by looking at the definition of $\otimes_{\mathfrak{C},R}$.

$$\begin{aligned} P_*(\not -) \otimes_{\mathfrak{C},\mathbb{Z}} \varphi^* A(\not -) &= \bigoplus_{x\in\mathfrak{C}} P_*(x) \otimes_{\mathbb{Z}} \varphi^* A(x) \Big/ \sim \\ &\cong \bigoplus_{x\in\mathfrak{C}} P_*(x) \otimes_{\mathbb{Z}} (R \otimes_R A(x)) \Big/ \sim \\ &\cong \bigoplus_{x\in\mathfrak{C}} (P_*(x) \otimes_{\mathbb{Z}} R) \otimes_R A(x) \Big/ \sim \\ &= (P_*(\not -) \otimes_{\mathbb{Z}} R) \otimes_{\mathcal{O}_{\mathcal{F}},R} A(\not -) \end{aligned}$$

For the general case, let $\varphi_1 : \mathbb{Z} \to R_1$ and $\varphi_2 : \mathbb{Z} \to R_2$ be the unique ring homomorphisms, then $\varphi \circ \varphi_1 = \varphi_2$ and $\varphi_1^* \circ \varphi^* = \varphi_2^*$. Applying the previous part twice

$$\begin{aligned} \mathrm{Tor}_*^{R_1,\mathfrak{C}}(\underline{R}_1(\not -), \varphi^* A(\not -)) &\cong \mathrm{Tor}_*^{\mathbb{Z},\mathfrak{C}}(\underline{\mathbb{Z}}(\not -), \varphi_1^* \circ \varphi^* A(\not -)) \\ &\cong \mathrm{Tor}_*^{R_2,\mathfrak{C}}(\underline{R}_2(\not -), A(\not -)) \end{aligned}$$

□

The following result is essentially [Ham08, 1.4.3], where is it proved for rings of prime characteristic in the setting of ordinary group cohomology.

Proposition 1.32 Given some integer $m > 0$ and ring R with characteristic m, then $\underline{R}(-)$ is $\mathfrak{C}\mathrm{FP}_n$ over R if and only if $\underline{\mathbb{Z}/m\mathbb{Z}}(-)$ is $\mathfrak{C}\mathrm{FP}_n$ over $\mathbb{Z}/m\mathbb{Z}$ (here $\underline{R}(-)$ and $\underline{\mathbb{Z}/m\mathbb{Z}}(-)$ are either both covariant or both contravariant modules).

Proof. The proof below is for contravariant modules, the proof for covariant modules is analogous.

Assume that $\underline{\mathbb{Z}/m\mathbb{Z}}(-)$ is $\mathfrak{C}\mathrm{FP}_n$ over $\mathbb{Z}/m\mathbb{Z}$. If $M_*(-)$ is any directed system of contravariant $\mathfrak{C}$-modules over R with $\varinjlim M_*(-) = 0$, we necessarily have $\varinjlim \varphi^* M_*(-) = 0$. By Theorem 1.28, and the fact that $\underline{\mathbb{Z}/m\mathbb{Z}}$ is assumed $\mathfrak{C}\mathrm{FP}_n$ over $\mathbb{Z}/m\mathbb{Z}$, we have that for all $i \leq n$,

$$\varinjlim \mathrm{Ext}^i_{\mathfrak{C},\mathbb{Z}/m\mathbb{Z}}(\underline{\mathbb{Z}/m\mathbb{Z}}(\not -), \varphi^* M_*(\not -)) = 0$$

Thus by Proposition 1.31 applied to the canonical map $\mathbb{Z}/m\mathbb{Z} \to R$,

$$\varinjlim \mathrm{Ext}^i_{\mathfrak{C},R}(\underline{R}(\not -), M_*(\not -)) = 0$$

Theorem 1.28 gives that $\underline{R}(-)$ is $\mathfrak{C}\mathrm{FP}_n$ over R.

For the "only if" direction, suppose $M_*(-)$ is a directed system of $\mathfrak{C}$-modules over $\mathbb{Z}/m\mathbb{Z}$, with $\varinjlim M_*(-) = 0$ thus $\varinjlim M_*(-) \otimes_{\mathbb{Z}/m\mathbb{Z}} R = 0$ and by Theorem 1.28 for all $i \leq n$,

$$\varinjlim \mathrm{Ext}^i_{\mathcal{O}_{\mathcal{F}},R}(\underline{R}(\not -), M_*(\not -) \otimes_{\mathbb{Z}/m\mathbb{Z}} R) = 0$$

Combining with Proposition 1.31

$$\varinjlim \operatorname{Ext}^i_{\mathbb{Z},\mathfrak{C}}(\underline{\mathbb{Z}/m\mathbb{Z}}(\not-), M_*(\not-) \otimes_{\mathbb{Z}/m\mathbb{Z}} R) = \varinjlim \operatorname{Ext}^i_{R,\mathfrak{C}}(\underline{R}(\not-), M_*(\not-) \otimes_{\mathbb{Z}/m\mathbb{Z}} R)$$
$$= 0$$

Since $\mathbb{Z}/m\mathbb{Z}$ is self-injective [Lam99, Cor 3.13], R splits as a $\mathbb{Z}/m\mathbb{Z}$ module into $R \cong \mathbb{Z}/m\mathbb{Z} \oplus N$ where N is some $\mathbb{Z}/m\mathbb{Z}$ module. Thus we have

$$\varinjlim \Big(\operatorname{Ext}^i_{\mathbb{Z}/m\mathbb{Z},\mathfrak{C}}(\underline{\mathbb{Z}/m\mathbb{Z}}(\not-), M_*(\not-)) \oplus \operatorname{Ext}^i_{\mathbb{Z}/m\mathbb{Z},\mathfrak{C}}(\underline{\mathbb{Z}/m\mathbb{Z}}(\not-), M_*(\not-) \otimes_{\mathbb{Z}/m\mathbb{Z}} N)\Big) = 0$$

In particular

$$\varinjlim \operatorname{Ext}^i_{\mathbb{Z}/m\mathbb{Z},\mathfrak{C}}(\underline{\mathbb{Z}/m\mathbb{Z}}(\not-), M_*(\not-)) = 0$$

So by Theorem 1.28 $\underline{\mathbb{Z}/m\mathbb{Z}}(-)$ is $\mathfrak{C}\mathrm{FP}_n$ over $\mathbb{Z}/m\mathbb{Z}$. □

Remark 1.33. This proposition fails in characteristic zero as the ring $\mathbb{Z}$ is not self-injective. For example $\mathbb{Q}$ is not isomorphic, as a $\mathbb{Z}$-module, to $N \otimes \mathbb{Z}$ for any $\mathbb{Z}$-module N.

2 Bredon Modules

Given a family $\mathcal{F}$ of subgroups of G, closed under subgroups and conjugation, recall from Example 1.10 the Orbit Category $\mathcal{O}_{\mathcal{F}}$ is the category whose objects are transitive G-sets and whose morphism set $[G/H, G/K]_{\mathcal{O}_{\mathcal{F}}}$ is the free abelian group on the set of G-maps $G/H \to G/K$. Since we will be dealing exclusively with the orbit category for much of this Section, we will write $[G/H, G/K]$ instead of $[G/H, G/K]_{\mathcal{O}_{\mathcal{F}}}$ when there is no possibility for confusion.

From now on we specialise to the family of all finite subgroups of G, setting $\mathcal{F} = \mathcal{F}in$. Many of the results remain true for arbitrary families, and this will be mentioned where possible.

Contravariant modules and their associated finiteness conditions are very well studied, as they provide a good algebraic setting to mirror the geometric world of proper actions. This background has already been discussed in the introduction. For additional information about the interplay of the geometry of proper actions and the finiteness conditions discussed in this section see [BLN01].

Sections 2.1, 2.2 and 2.3 will specialise information from Section 1 to modules over the orbit category, and the later sections will discuss finiteness conditions over the category of covariant and contravariant modules over the orbit category.

In Section 1, the categories $\mathfrak{C}$ considered were not assumed to have property (EI), see Remark 1.1. The first important observation is that the orbit category $\mathcal{O}_{\mathcal{F}}$ does have (EI), since any G-map $\alpha : G/K \to G/K$ is automatically an automorphism. The first task is to determine the internal structure of the category $\mathcal{O}_{\mathcal{F}}$.

Remark 2.1. Morphisms in $\mathcal{O}_{\mathcal{F}}$.

A G-map $\alpha : G/H \longrightarrow G/K$ is completely determined by the element $\alpha(H) = gK$, and such an element $gK \in G/K$ determines a G-map if and only if $HgK = gK$, usually written as $gK \in (G/K)^H$. The identification $R[G/H, G/K] \cong R[(G/K)^H]$ will be used freely from now on.

Equivalently an element gK determines a G-map $\alpha : H \mapsto gK$ if and only if $g^{-1}Hg \leq K$. Notice that the isomorphism classes of elements in $\mathcal{O}_{\mathcal{F}}$, denoted $\operatorname{Iso}\mathcal{O}_{\mathcal{F}}$, are exactly the conjugacy classes of subgroups in $\mathcal{F}$.

Remark 2.2. Structure of $\operatorname{Aut}(G/H)$.

If

$$\begin{aligned} \alpha_g : G/H &\longrightarrow G/H \\ H &\longmapsto gH \end{aligned}$$

is any G-map then such an α_g determines a G-map if and only if $g \in WH = N_GH/H$. Furthermore $\alpha_h \circ \alpha_g = \alpha_{gh}$, so combining these two pieces of information,

$$\operatorname{Aut}(G/H) = WH^{\mathrm{op}}$$

As described in Remark 1.6, if $A(-)$ is a covariant $\mathfrak{C}$-module then evaluating at x gives $A(x)$ an $R\operatorname{End}(x)$-structure. Thus evaluating a covariant Bredon module at gives a left $R[WH^{\mathrm{op}}]$-structure, equivalently a *right* $R[WH]$-structure.

Similarly, evaluating a contravariant Bredon module $M(-)$ at G/H gives $M(G/H)$ a left $R[\mathrm{Aut}(G/H)] \cong R[WH^{\mathrm{op}}]$ structure, equivalently a *left* $R[WH]$-module structure. This reversing of left and right structures is unfortunate, it would be possible to treat $A(G/H)$ as a left $R[WH]$-module via the map $g \mapsto g^{-1}$ and similarly for $M(G/H)$, but we choose not to do this as it makes the notation more confusing when we are dealing with the action on free modules, for example when we compute the right action of $R[WH]$ on $R[G/K, -]_{\mathcal{O}_\mathcal{F}}(G/H) = R[G/K, G/H]_{\mathcal{O}_\mathcal{F}}$ in Example 2.3.

2.1 Free Modules

In this section we describe the structure of free Bredon modules. Throughout this section H and K will denote finite subgroups of a group G. In fact, all the results in this section remain true over arbitrary families of subgroups, except for Corollary 2.7 and Lemma 2.8.

Example 2.3. Right action of $R[WK]$ on $R[G/H, -](G/K) = R[G/H, G/K]$
The action of WK on $R[G/H, G/K]$ is as follows: If $f \in R[G/H, G/K]$ with $f(H) = gK$ and $w \in WK$ then

$$f \cdot w = R[G/H, \alpha_w](f) = \alpha_w \circ f$$

Since $(\alpha_w \circ f)(1) = gwK$, under the identification $R[G/H, G/K] \cong R[(G/K)^H]$, the action is given by $gK \cdot w = gwK$.

Lemma 2.4 There is an isomorphism of right $R[WK]$-modules

$$R[G/H, -](G/K) = R[G/H, G/K] \cong \bigoplus_{\substack{gN_GK \in G/N_GK \\ g^{-1}Hg \leq K}} R[WK]$$

Proof. Firstly, $R[G/H, G/K] \cong R[(G/K)^H]$ is a free WK-module, since if $n \in N_GK$ such that $gnK = gK$ then $nK = K$ and hence $n \in K$. Now, gK and $g'K$ lie in the same WK orbit if and only if $g(WK)K = g'(WK)K$, equivalently $gN_GK = g'N_GK$, and gK determines an element of $R[(G/K)^K]$ if and only if $g^{-1}Hg \leq K$. Thus there is one $R[WK]$ orbit for each element in the set

$$\{gN_GK \in G/N_GK \ : \ g^{-1}Hg \leq K\}$$

□

For contravariant modules the situation is more complex, evaluating at G/H doesn't always give a free $R[WH]$-module, although it does always give a $R[WH]$-module of type FP_∞. This is proved in the case $R = \mathbb{Z}$ in [KMPN09, Proof of 3.2], the proof for general rings R requires no substantial change, and is given in Corollary 2.7.

Example 2.5. Left action of $R[WH]$ on $R[-, G/K](G/H) = R[G/H, G/K]$. A similar argument to the previous example shows that under the identification $R[G/H, G/K] \cong R[(G/K)^H]$, the action of $R[WH]$ is given by $w \cdot gK = wgK$.

Lemma 2.6 There is an isomorphism of left $R[WH]$-modules:

$$R[-, G/K](G/H) = R[G/H, G/K] = \bigoplus_x R[WH/WH_{xK}]$$

Where x runs over a set of coset representatives of the subset of the set of $N_G H$-K double cosets.

$$\{x \in N_G H \backslash G/K \; : \; x^{-1}Hx \leq K\}$$

and the stabilisers are given by

$$WH_{xK} = \left(N_G H \cap xKx^{-1}\right)/H$$

Proof. Using the identification $[G/H, G/K] = (G/K)^H$, the elements xK and yK are in the same WH-orbit if there exists some $nH \in WH$ (where $n \in N_G H$) with

$$nHxK = yK \Leftrightarrow nxK = yK \Leftrightarrow (N_G H)xK = (N_G H)yK$$

Combining this with the fact that $xK \in (G/K)^H$ if and only if $x^{-1}Hx \leq K$ means there is a WH-orbit for each $N_G H$-K double coset $N_G HxK$ such that $x^{-1}Hx \leq K$, ie coset representatives for

$$\{x \in N_G H \backslash G/K \; : \; x^{-1}Hx \leq K\}$$

are orbit representatives for the WH-orbits in $[G/H, G/K]$.

The $N_G(H)$-stabiliser of the point $xK \in (G/K)^H$ is the set

$$\{g \in N_G(H) \; : \; gxK = xK\} = \{g \in N_G(H) \; : \; g \in xKx^{-1}\} = N_G(H) \cap xKx^{-1}$$

So the WH-stabiliser of $xK \in (G/K)^H$ is $WH_{xK} = (N_G(H) \cap xKx^{-1})/H$. □

Corollary 2.7 $R[-, G/K](G/H) = R[G/H, G/K]$ is a finite direct sum of projective $R[WH]$-permutation modules of type FP_∞ with finite stabilisers. In particular $R[G/H, G/K]$ is FP_∞.

Proof. Since K is finite, the set $\{x \in N_G H \backslash G/K \; : \; x^{-1}Hx \leq K\}$ is finite and $R[WH]$ can be written as a finite direct summand

$$R[G/H, G/K] = \bigoplus_x R[WH/WH_{xK}]$$

WH_{xK} is a finite group and as such R is FP_∞ as a $R[WH_{xK}]$-module.

$$R[WH/WH_{xK}] = \mathrm{Ind}_{R[WH_{xK}]}^{R[WH]} R$$

We apply Lemma 2.8 below and deduce that $R[WH/WH_{xK}]$ is FP_∞ as a RG-module. Finally, any finite direct sum of FP_∞ modules is FP_∞. □

Lemma 2.8 If M is FP_∞ as an RF-module for some subgroup $F \leq G$, then $\mathrm{Ind}_{RF}^{RG} M = RG \otimes_{RF} M$ is FP_∞ as an RG-module.

Proof. Let $\prod_i N_i$ be an arbitrary direct product of RG-modules, then

$$\begin{aligned}\mathrm{Tor}_*^{RG}\left(\mathrm{Ind}_{RF}^{RG} M, \prod_i N_i\right) &= \mathrm{Tor}_*^{RF}\left(M, \prod_i N_i\right) \\ &= \prod_i \mathrm{Tor}_*^{RF}(M, N_i) \\ &= \prod_i \mathrm{Tor}_*^{RG}\left(\mathrm{Ind}_{RF}^{RG} M, N_i\right)\end{aligned}$$

where the first and third equalities come from Shapiro's Lemma. This finishes the proof as $\mathrm{Ind}_{RF}^{RG} M$ is FP_∞ if and only if $\mathrm{Tor}_*^{RG}(\mathrm{Ind}_{RF}^{RG} M, -)$ commutes with direct products [Bro94, Theorem VIII.4.8]. □

2.2 Restriction, Induction and Coinduction

We specialise the constructions of Section 1.3 to the categories of covariant and contravariant Bredon modules. We write $\mathrm{Ind}_H^{\mathcal{O}_\mathcal{F}} A$ instead of $\mathrm{Ind}_{R\,\mathrm{Aut}(G/H)}^{\mathcal{O}_\mathcal{F}} A$, and similarly for restriction and coinduction.

Example 2.9. If R is the trivial RG module then

$$\mathrm{Ind}_1^{\mathcal{O}_\mathcal{F}^{\mathrm{cov}}} R(-) : G/H \mapsto R \otimes_{RG} R[G/H] = R$$

Checking the morphisms as well, $\mathrm{Ind}_1^{\mathcal{O}_\mathcal{F}^{\mathrm{cov}}} R(-) = \underline{R}(-)$, the constant covariant functor on R.

A group is said to *contain no R-torsion* if for every finite subgroup $F \leq G$, $|F|$ is invertible in R. For example every group has no $\mathbb{Q}$-torsion. If

$$|F| = p_1^{n_1} \cdots p_m^{n_m}$$

is a prime factorisation of $|F|$ then for each p_i there is an element of order p_i by Cauchy's Theorem. [Rob96, 1.6.17] Since the invertible elements R^* form a group, if all the p_i are invertible in R then so is $|F|$. Hence a group has no R-torsion if and only if the order of every finite-order *element* is invertible in R.

Recall from Proposition 1.20 that covariant and contravariant restriction is exact, in addition we have the following:

Proposition 2.10
1. Covariant restriction preserves projectives and flats.
2. Contravariant restriction preserves finite generation.
3. Contravariant restriction at H preserves projectives and flats if WH is R-torsion-free, if not then contravariant restriction takes projectives to FP_∞-modules.

Proof.
1. If $P(-)$ is a projective covariant Bredon module and $F(-)$ a free covariant Bredon module with a split epimorphism $F(-) \twoheadrightarrow P(-)$ then

restricting at G/H yields a split epimorphism $F(G/H) \twoheadrightarrow P(G/H)$, by Lemma 2.4 $F(G/H)$ is free and thus $P(G/H)$ is projective.

If $F(-)$ is a flat covariant module and M any left $R[WH]$-module then,

$$\begin{aligned} F(G/H) \otimes_{R[WH]} M &\cong (R[\not -, G/H] \otimes_{\mathcal{O}_\mathcal{F}} F(\not -)) \otimes_{R[WH]} M \\ &\cong \big(R[\not -, G/H] \otimes_{R[WH]} M\big) \otimes_{\mathcal{F}} F(\not -) \end{aligned}$$

Where the second isomorphism is Remark 1.17.

Thus for any short exact sequence of left $R[WH]$-modules.

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

Applying $F(G/H) \otimes_{R[WH]} -$ is equivalent to applying first the contravariant induction functor and then $\dagger \otimes_{\mathcal{F}} F(\not -)$. Since contravariant induction is exact (Proposition 2.13(2)) and $F(-)$ is assumed flat, exactness is preserved, and thus $F(G/H)$ is flat as required.

2. Use the argument of the previous part, noting that Lemma 2.6 implies that for contravariant frees, unlike for covariant frees, restricting at G/H preserves finite generation.
3. If WH is R-torsion-free then, using Lemma 2.6, restricting any free at G/H gives a projective module, and the result follows. To see that in this case, restriction preserves flats, let $F(-)$ be a contravariant flat module and consider a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

of left $R[WH]$-modules, thus by Proposition 2.13 below,

$$0 \longrightarrow \mathrm{Ind}_H^{\mathcal{O}_\mathcal{F}^{\mathrm{cov}}} A(-) \longrightarrow \mathrm{Ind}_H^{\mathcal{O}_\mathcal{F}^{\mathrm{cov}}} B(-) \longrightarrow \mathrm{Ind}_H^{\mathcal{O}_\mathcal{F}^{\mathrm{cov}}} C(-) \longrightarrow 0$$

is a short exact sequence of covariant modules. Since $F(-)$ is flat, the functor $\dagger \otimes_{\mathcal{O}_\mathcal{F}} F(\not -)$ is exact, applying this to the above and using Lemma 1.21 gives a short exact sequence

$$0 \longrightarrow A \otimes_R F(G/H) \longrightarrow B \otimes_R F(G/H) \longrightarrow C \otimes_R F(G/H) \longrightarrow 0$$

Showing $F(G/H)$ is flat.

If WH is not R-torsion free then the result is just Corollary 2.7.

□

Example 2.11. Unlike in the contravariant case, the covariant restriction functor does not preserve "finitely generated" in general: Take for example the infinite dihedral group $D_\infty = \mathbb{Z}_2 * \mathbb{Z}_2$ generated by the two elements a and b of order 2. The finite subgroup $\langle a \rangle$ is self-normalising, thus $R[W\langle a \rangle] = R$ and Lemma 2.4 implies that as R-modules

$$R[D_\infty/1, D_\infty/\langle a \rangle] = \bigoplus_{g\langle a \rangle \in D_\infty/\langle a \rangle} R$$

Remark 2.12. The functor $\mathrm{Res}_1^{\mathcal{O}_\mathcal{F}^{\mathrm{cov}}}$ preserves "finitely generated". Recall that

$$R[G/K, G/1] \cong \begin{cases} RG & \text{if } K = 1 \\ 0 & \text{else.} \end{cases}$$

So if $A(-)$ is an arbitrary finitely generated covariant Bredon module and $F(-)$ a free covariant Bredon module with an epimorphism onto $A(-)$ then $F(G/1)$ is finitely generated as an RG-module and since Res_1 is exact there is a surjection $F(G/1) \twoheadrightarrow A(G/1)$.

Recall from Proposition 1.20 that contravariant and covariant induction both preserve projectives, flats and finitely generation. In addition we have the following facts, which will play a crucial role in analysing finiteness conditions for covariant Bredon modules in Sections 2.4 and 2.5.

Proposition 2.13
1. If WH has no R-torsion the covariant induction functor $\mathrm{Ind}_H^{\mathcal{O}_\mathcal{F}^{\mathrm{cov}}}$ is exact.
2. Contravariant induction is always exact.

Proof.
1. Assume that WH has no R-torsion, we must check that the functor
$$A \longmapsto A \otimes_{R[WH]} R[G/H, -]$$
is exact, where A is an $R[WH]$-module. Equivalently that for any finite subgroup K of G, the functor
$$- \otimes_{R[WH]} R[G/H, G/K]$$
is exact, but by Lemma 2.6
$$R[G/H, G/K] = \bigoplus_{x \in I} R\left[WH/WH_x\right]$$
For some finite indexing set I and WH_x finite subgroups of WH. By Maschke's Theorem, $R\left[WH/WH_x\right]$ is projective, and hence flat, as an $R[WH]$ module. Hence $- \otimes_{R[WH]} R[G/H, G/K]$ is indeed exact.
2. Similarly to the above, we must check the functor
$$R[G/K, G/H] \otimes_{R[WH]} -$$
is exact, but by Lemma 2.4, $R[G/K, G/H]$ is free as an $R[WH]$-module so this is automatic.

□

In summary:

Covariant restriction is exact and preserves both projectives and flats. Covariant restriction at $G/1$ preserves finite generation.	Contravariant restriction is exact and preserves finitely generated, the restriction at G/H of a projective is projective if WH is R-torsion-free, else it is FP_∞.
Covariant induction at H is right exact and preserves projectives, flats and finite generation. If WH is R-torsion-free, covariant induction is exact.	Contravariant induction at H is exact and preserves projectives, flats and finite generation.
Covariant coinduction at H preserves injectives and is left exact.	Contravariant coinduction at H preserves injectives and is left exact.

2.3 Covariant Homology and Cohomology

We make the following definitions for any contravariant Bredon module $M(-)$, any covariant Bredon module $A(-)$, and $\underline{R}(-)$ the constant covariant Bredon module.

$$\begin{aligned}\text{cov-}\mathrm{H}_*^{\mathcal{O}_\mathcal{F}}(G, M(\not{-})) &= \mathrm{Tor}_*^{\mathcal{O}_\mathcal{F}}(M(\not{-}), \underline{R}(\not{-}))\\ \text{cov-}\mathrm{H}^*_{\mathcal{O}_\mathcal{F}}(G, A(\not{-})) &= \mathrm{Ext}^*_{\mathcal{O}_\mathcal{F}}(\underline{R}(\not{-}), A(\not{-}))\end{aligned}$$

Note that the $\mathrm{Ext}^*_{\mathcal{O}_\mathcal{F}}$ above is taken with two covariant modules, in contrast to the usual usage with two contravariant modules.

Proposition 2.14 For any contravariant Bredon module $M(-)$ and covariant Bredon module $A(-)$.

1. $\text{cov-}\mathrm{H}_*^{\mathcal{O}_\mathcal{F}}(G, M(\not{-})) = H_*(G, M(G/1))$.
2. If G has no R-torsion then $\text{cov-}\mathrm{H}^*_{\mathcal{O}_\mathcal{F}}(G, A(\not{-})) = H^*(G, A(G/1))$.

Proof. 1. Let F_* be a resolution of R by flat right RG-modules, then by Proposition 1.20 and Example 2.9 $\mathrm{Ind}_1^{\mathcal{O}_\mathcal{F}^{\mathrm{cov}}} F_*(-)$ is a resolution of $\underline{R}(-)$ by flat covariant Bredon modules. Applying $M(-) \otimes_{\mathcal{O}_\mathcal{F}} -$ yields the resolution

$$\begin{aligned}M(\not{-}) \otimes_{\mathcal{O}_\mathcal{F}} (F_* \otimes_{RG} R[G/1, \not{-}]) &\cong F_* \otimes_{RG} (M(\not{-}) \otimes_{\mathcal{O}_\mathcal{F}} R[G/1, \not{-}])\\ &\cong F_* \otimes_{RG} M(G/1)\end{aligned}$$

Where the above two natural isomorphisms are from Remark 1.17 and the Yoneda-type Lemma (1.5). Finally, since homology can be calculated from a flat resolution [Rot09, 7.5],

$$\begin{aligned}\text{cov-}\mathrm{H}_*^{\mathcal{O}_\mathcal{F}}(G, M(\not{-})) &\cong H_*\big(M(-) \otimes_{\mathcal{O}_\mathcal{F}} (P_* \otimes_{RG} R[G/1, \not{-}])\big)\\ &\cong H_*\big(P_* \otimes_{RG} M(G/1)\big)\\ &= H_*(G, M(G/1))\end{aligned}$$

2. Let P_* be a resolution of R by projective right RG-modules, by Proposition 1.20 and Example 2.9 $\mathrm{Ind}_1^{\mathcal{O}_\mathcal{F}^{\mathrm{cov}}} P_*(-)$ is a resolution of $\underline{R}(-)$ by projective covariant modules. Apply $\mathrm{Mor}_{\mathcal{O}_\mathcal{F}}(-, A(\not{-}))$ to get the resolution

$$\mathrm{Mor}_{\mathcal{O}_\mathcal{F}}(E_1 P_*(\not{-}), A(\not{-})) \cong \mathrm{Hom}_{RG}(P_*, A(G/1))$$

The isomorphism is the adjoint isomorphism between induction and restriction. Thus

$$\begin{aligned}\text{cov-H}^*_{\mathcal{O}_\mathcal{F}}(G, A(\not-)) &\cong H^* \operatorname{Mor}_{\mathcal{O}_\mathcal{F}}(\operatorname{Ind}_1^{\mathcal{O}_\mathcal{F}^{\text{cov}}} P_*(\not-), A(\not-)) \\ &\cong H^* \operatorname{Hom}_{RG}(P_*, A(G/1)) \\ &\cong H^*(G, A(G/1))\end{aligned}$$

□

2.4 Covariant Cohomological Dimension

This section is devoted to an analysis of finiteness conditions for covariant modules. Fix an orbit category $\mathcal{O}_\mathcal{F}$ and ring R. Recall from Section 1.5 that a group G has $\mathcal{O}_\mathcal{F}^{\text{cov}} \operatorname{cd}_R G \leq n$, or *covariant cohomological dimension* less than n if the constant functor $\underline{R}(-)$ has $\mathcal{O}_\mathcal{F}^{\text{cov}} \operatorname{cd}_R \underline{R}(-) \leq n$. Similarly for covariant homological dimension. The covariant cohomological dimension and covariant homological dimension are easy to classify.

Theorem 2.15
1. The conditions covariant-$\mathcal{O}_\mathcal{F}^{\text{cov}} \operatorname{cd}_R G \leq n$ and $\operatorname{cd}_R G \leq n$ are equivalent.
2. The conditions covariant-$\mathcal{O}_\mathcal{F}^{\text{cov}} \operatorname{hd}_R G \leq n$ and $\operatorname{hd}_R G \leq n$ are equivalent.

Proof.
1. If G satisfies $\mathcal{O}_\mathcal{F}^{\text{cov}} \operatorname{cd}_R G \leq n$ then, by Proposition 2.10, if $P_*(-)$ is a length n projective resolution of $\underline{R}$ by projective $\mathcal{O}_\mathcal{F}$-modules then $P_*(G/1)$ is a length n projective resolution of R by projective RG-modules and thus $\operatorname{cd}_R G \leq n$

   For the converse, note first that $\operatorname{cd}_R G \leq n$ implies that G has no R-torsion. Pick a length n projective resolution of R by projective RG-modules and consider $(\operatorname{Ind}_1^{\mathcal{O}_\mathcal{F}^{\text{cov}}} P_*)(-)$, this is a resolution of $\underline{R}(-)$ by projective Bredon modules by Propositions 2.13, 1.20 and Example 2.9.
2. This is proved exactly as in the previous case. If G satisfies $\mathcal{O}_\mathcal{F}^{\text{cov}} \operatorname{hd}_R G \leq n$ then take a length n flat resolution $F_*(-)$ of $\underline{R}(-)$, by Propositions 1.20 and 2.10 and Example 2.9, $F_*(G/1)$ is a length n flat resolution of R by RG-modules. For the converse take a finite flat resolution of R by RG-modules, apply the extension functor $\operatorname{Ind}_1^{\mathcal{O}_\mathcal{F}^{\text{cov}}}$ and use Proposition 1.20 and Example 2.9 again.

□

2.5 Covariant FP_n Conditions

This section contains two observations about the covariant $\mathcal{O}_\mathcal{F} \text{FP}_n$ conditions. Recall that a group G has covariant-$\mathcal{O}_\mathcal{F} \text{FP}_n$ over a ring R if there is a resolution of the constant functor $\underline{R}(-)$ by projective covariant $\mathcal{O}_\mathcal{F}$-modules, finitely generated up to dimension n.

Theorem 2.16 If G is covariant-$\mathcal{O}_\mathcal{F} \text{FP}_n$ then G is FP_n, if G has no R-torsion and is FP_n then G is covariant $\mathcal{O}_\mathcal{F} \text{FP}_n$.

Proof. For the "only if" part, take a projective resolution $P_*(-)$ of $\underline{R}$ by covariant projective modules, finitely generated up to dimension n, apply Proposition 2.10 and Remark 2.12 to get that $P_*(G/1)$ is a projective resolution of R by projective RG-modules, finitely generated up to dimension n.

For the "if" part, choose a projective resolution P_* of R by projective RG-modules, finitely generated up to dimension n. Then the induced resolution $\mathrm{Ind}_1^{\mathcal{O}_\mathcal{F}^{\mathrm{cov}}} P_*(-)$ is a resolution of $\underline{R}(-)$ by projective covariant Bredon modules, finitely generated up to dimension n by Proposition 2.13 and Example 2.9. □

Proposition 2.17

1. Every group G is covariant-$\mathcal{O}_\mathcal{F}\,\mathrm{FP}_0$ over R.
2. G is covariant-$\mathcal{O}_\mathcal{F}\,\mathrm{FP}_1$ over R if and only if G is FP_1 over R if and only if G is finitely generated.

Proof.

1. The augmentation map $R[G/1,-] \longrightarrow \underline{R}(-)$ is an epimorphism.
2. Choose finitely generated projective RG-modules P_0, P_1 with an exact sequence
$$P_1 \longrightarrow P_0 \longrightarrow R \longrightarrow 0 \qquad (*)$$
The induction functor is always right exact, preserves projectives and preserves "finitely generated" (Proposition 1.20). Finally Example 2.9 shows $\mathrm{Ind}_1^{\mathcal{O}_\mathcal{F}^{\mathrm{cov}}} R(-) \cong \underline{R}(-)$, so applying $\mathrm{Ind}_1^{\mathcal{O}_\mathcal{F}^{\mathrm{cov}}}$ to $(*)$ completes the proof.

□

Question 2.18. Is there a nice characterisation of the condition covariant-$\mathcal{O}_\mathcal{F}\,\mathrm{FP}_n$ over R, for groups which are not R-torsion free?

2.6 Contravariant Cohomological Dimension

This subsection lists some well-known results concerning Bredon cohomological dimension over arbitrary rings, with proofs given for results not easily available in the literature.

Lemma 2.19

1. If $\mathrm{cd}_\mathbb{Z} G \leq n$ then $\mathrm{cd}_R G \leq n$ for all rings R.
2. IF $\mathcal{O}_\mathcal{F}\,\mathrm{cd}_\mathbb{Z} G \leq n$ then $\mathcal{O}_\mathcal{F}\,\mathrm{cd}_R G \leq n$ for all rings R.

Proof.

1. Take a projective resolution P_* of $\mathbb{Z}$ by $\mathbb{Z}G$ modules of length n, it is acyclic over $\mathbb{Z}$ and hence $\mathbb{Z}$-split. $P_* \otimes_\mathbb{Z} R$ is a projective resolution of R by RG-modules of length n.
2. Take a projective resolution of $\underline{\mathbb{Z}}$ by contravariant modules of length n, define a new resolution by $Q_n(G/H) = P_n(G/H) \otimes_\mathbb{Z} R$ for all $n \in \mathbb{N}$ and $G/H \in \mathcal{O}_\mathcal{F}$. The tensor product here is the tensor product of Section 1.1.2.
$$- \otimes_\mathbb{Z} R : \{\mathcal{O}_\mathcal{F}\text{-modules}\} \to \{\mathcal{O}_\mathcal{F}\text{-modules}\}$$
$$\big(M(-) \otimes_\mathbb{Z} R\big)(G/H) = M(G/H) \otimes_\mathbb{Z} R$$
By the argument of the previous part, $Q_*(G/H)$ is acyclic for all G/H and so Q_* is acyclic. Finally we show that each Q_n is projective as an

contravariant module over R: Since P_n is projective there is a split short exact sequence

$$0 \longrightarrow K \longrightarrow \bigoplus_{i\in I} \mathbb{Z}[-, G/H_i] \longrightarrow P_n \longrightarrow 0$$

where I is some index set and H_i is a finite subgroup for all i. Since

$$\left(\bigoplus_{i\in I} \mathbb{Z}[-, G/H_i](G/H)\right) \otimes_{\mathbb{Z}} R = \left(\bigoplus_{i\in I} R[-, G/H_i]\right)(G/H)$$

there is a split short exact sequence

$$0 \longrightarrow K' \longrightarrow \bigoplus_{i\in I} R[-, G/H_i] \longrightarrow Q_n \longrightarrow 0$$

and Q_n is a projective contravariant module over R.

□

We'll need the following two well known lemmas.

Lemma 2.20 [Bie81, Proposition 4.11] If $\mathrm{cd}_R\, G \leq n$ for some $n \in \mathbb{N}$ then G has no R-torsion.

Lemma 2.21 [Bie81, Propisition 4.12] $R[G/H]$ is a projective RG-module if and only if $|H|$ is finite and invertible in R.

Lemma 2.22 For any ring R, if G has no R-torsion then $\mathrm{cd}_R\, G \leq \mathcal{O}_{\mathcal{F}}\, \mathrm{cd}_R\, G$.

Proof. Take a projective resolution of contravariant modules over R of length n and evaluate at $G/1$, since G is R-torsion-free, Proposition 2.10 implies that $P_*(G/1)$ is a length n projective resolution of $\mathbb{Z}$ by $\mathbb{Z}G$-modules. □

There is the following generalisation of Serre's theorem for R-torsion free groups (see [Bro94, VIII.3] for the classical case).

Theorem 2.23 [Coh72, p.9 Theorem C] If R is commutative, G has no R-torsion and H is finite index in G then $\mathrm{cd}_R\, H = \mathrm{cd}_R\, G$.

2.6.1 Low Dimensions

We classify those groups with $\mathcal{O}_{\mathcal{F}}\, \mathrm{cd}_R\, G = 0$ and $\mathcal{O}_{\mathcal{F}}\, \mathrm{cd}_{\mathbb{Z}}\, G = 1$.

Proposition 2.24 $\underline{R}(-)$ is projective if and only if G is finite.

An alternative proof of this is available in [Flu10, Prop 3.20], which is based around a result in [Sym10]. We give a proof from first principles.

Proof. Assume $\underline{R}(-)$ is projective and let H_i be some collection of finite subgroups such that there is a split surjection

$$\bigoplus_i R[-, G/H_i] \overset{\oplus\pi_i}{\twoheadrightarrow} \underline{R}(-)$$

Denote the splitting by $\oplus_i s : \underline{R}(-) \to \oplus_i R[-, G/H_i]$, where each s_i is the map $\underline{R}(-) \to R[-, G/H_i]$. Consider $\oplus\pi_i(G/1) : \oplus R[G/1, G/H_i] \to R$, the splitting of this map must factor through only one factor of $\oplus R[G/1, G/H_i]$, denote this factor $R[G/1, G/H_1]$. In other words $s_i(G/1) \neq 0$ if and only if $i = 1$. The commutative diagram representing $\oplus_i s_i$ as a natural transformation looks as follows (except here we're only showing one finite subgroup K of G, and one G-map $\alpha : G/1 \to G/K$).

$$\begin{array}{ccc} R & \xrightarrow{\oplus_i s_i(G/K)} & \bigoplus_i R[G/K, G/H_i] \\ {\scriptstyle \alpha_* = \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \alpha_*} \\ R & \xrightarrow{\oplus_i s_i(G/1)} & \bigoplus_i R[G/1, G/H_i] \end{array}$$

If $s_i(G/K) \neq 0$ for some $i \neq 1$, then $s_i(G/1) \neq 0$ by commutativity, leading to a contradiction. Hence $s_i \neq 0$ if and only if $i = 1$ and we have a split surjection:

$$R[-, G/H_1] \overset{\pi}{\twoheadrightarrow} \underline{R}(-)$$

Assume G is not finite, evaluating π at $G/1$ gives a split surjection $R[G/H_1] \longrightarrow R$, but this is impossible since G/H_1 is infinite. Hence G is finite.

For the converse, observe there is a unique map $G/H \to G/G$ and so the ordinary augmentation map

$$\varepsilon : R[-, G/G] \longrightarrow \underline{R}(-)$$

$$\varepsilon(G/H) : f \mapsto 1$$

is a surjection. □

This is an interesting contrast to the result that $\mathrm{cd}_R\, G = 0$ if and only if G is finite with no R-torsion [Bie81, Proposition 4.12]

Lemma 2.25 For any group G, $\mathcal{O}_{\mathcal{F}}\, \mathrm{cd}_{\mathbb{Z}}\, G = 0$ if and only if $\mathrm{cd}_{\mathbb{Q}}\, G = 0$ and $\mathcal{O}_{\mathcal{F}}\, \mathrm{cd}_{\mathbb{Z}}\, G = 1$ if and only if $\mathrm{cd}_{\mathbb{Q}}\, G = 1$.

Proof. Lemma 2.21 with $H = G$ implies that $\mathrm{cd}_{\mathbb{Q}}\, G = 0$ if and only if G is finite. Combining this with Proposition 2.24 we see that $\mathrm{cd}_{\mathbb{Q}}\, G = 0$ if and only if $\mathcal{O}_{\mathcal{F}}\, \mathrm{cd}_{\mathbb{Z}}\, G = 0$.

If $\mathcal{O}_{\mathcal{F}}\, \mathrm{cd}_{\mathbb{Z}}\, G = 1$ then Lemma 2.19 implies $\mathcal{O}_{\mathcal{F}}\, \mathrm{cd}_{\mathbb{Q}}\, G \leq 1$ and Lemma 2.22 implies $\mathrm{cd}_{\mathbb{Q}}\, G \leq 1$. Since G is not finite, $\mathrm{cd}_{\mathbb{Q}}\, G = 1$.

If $\mathrm{cd}_{\mathbb{Q}}\, G = 1$ then by [Dun79, Theorem 1.1], G acts properly and with finite stabilisers on a tree T. For any finite subgroup $H \leq G$, H acts on T, $T^H \neq \emptyset$ and in particular T^H is a sub-tree of T. [Ser03, 6.1, 6.3.1] T is thus a model for $\mathrm{E}_{\mathcal{F}in} G$ and $\mathcal{O}_{\mathcal{F}}\, \mathrm{cd}_{\mathbb{Z}}\, G = 1$. □

Corollary 2.26 The following are equivalent for an infinite group G, and any ring R:

1. $\operatorname{cd}_R G = 1$.
2. G has no R-torsion and acts properly on a tree.
3. G has no R-torsion and $\mathcal{O}_{\mathcal{F}} \operatorname{cd}_{\mathbb{Z}} G = 1$.
4. G has no R-torsion and $\mathcal{O}_{\mathcal{F}} \operatorname{cd}_R G = 1$.
5. G has no R-torsion and $\operatorname{cd}_{\mathbb{Q}} G = 1$.

Proof. $1 \Rightarrow 2$ If $\operatorname{cd}_R G = 1$ then G has no R-torsion by Lemma 2.20 and by [Dun79, Theorem 1.1] G acts properly on a tree.

$2 \Rightarrow 3$ If G acts properly on a tree then by the argument of Lemma 2.25 the tree is a model for $\mathrm{E}_{\mathcal{F}in} G$ and hence $\mathcal{O}_{\mathcal{F}} \operatorname{cd}_{\mathbb{Z}} G = 1$.

$3 \Rightarrow 4$ Lemma 2.19(2).

$4 \Rightarrow 1$ Lemma 2.22.

$3 \Leftrightarrow 5$ Lemma 2.25.

□

Question 2.27. What does the condition $\mathcal{O}_{\mathcal{F}} \operatorname{cd}_R G = 1$ represent? Is it equivalent to $\mathcal{O}_{\mathcal{F}} \operatorname{cd}_{\mathbb{Z}} G = 1$?

2.6.2 Some Interesting Examples

Example 2.28. A group with $\operatorname{cd}_{\mathbb{F}_3} G = 2$ and $\operatorname{cd}_{\mathbb{Z}} G = 3$.

See [Dav08, 8.5.8], although this example first appeared in [Bes93]. The torsion-free subgroup G of the right-angled Coxeter group (W, S) corresponding to the barycentric subdivision L of the ordinary triangulation of $\mathbb{RP}^2$ is shown to have $\operatorname{cd}_{\mathbb{Z}} G = 3$ and $\operatorname{cd}_{\mathbb{Q}} G = 2$. For an explanation of the notation used here see [Dav08]. By the right angled Coxeter group corresponding to L we mean the group W generated by a set S of involutions where S is in bijection with the vertices of L and two involutions commute if and only if they are adjancent in L. We use essentially the same argument as that on p.154 of [Dav08].

Using Davis' notation: If $\mathcal{S}$ is the poset of spherical subsets of S then let $\partial K = |\mathcal{S}_{>\emptyset}|$ and form $\mathcal{U}(W, \partial K)$. (This is different from the usual construction where we take $K = |S|$ and consider $\mathcal{U}(W, K)$ instead). We wish to show that $\mathcal{U}(W, \partial K)$ is $\mathbb{F}_3$-acyclic. [Dav08, 8.2.8] goes through with arbitrary coefficients: $\mathcal{U}(W, \partial K)$ is $\mathbb{F}_3$-acyclic if and only if $(\partial K)_T$ is $\mathbb{F}_3$-acyclic for all spherical subsets $T \in \mathcal{S}$. Recall that K_T denotes the intersection of mirrors $\cap_{s \in T} K_s$, where a mirror K_s is $|\mathcal{S}_{\geq s}|$.

If $T \neq \emptyset$ then $(\partial K)_T = K_T$ which is contractible and hence $\mathbb{F}_3$-acyclic and if $T = \emptyset$ then $(\partial K)_T = \partial K$ which is the barycentric subdivision of $L = \mathbb{RP}^2$ and hence $\mathbb{F}_3$-acyclic. Thus the torsion-free subgroup G of finite index in W acts freely on an $\mathbb{F}_3$-acyclic space $\mathcal{U}(W, \partial K)$ and satisfies $\operatorname{cd}_{\mathbb{F}_3} G \leq 2$.

Recall [Bie81, Corollary 3.6]: If R is hereditary and G is FP_∞ over R and L is any R-module, we have a short exact sequence

$$0 \longrightarrow H^q(G, RG) \otimes_R L \longrightarrow H^q(G, L \otimes_R RG) \longrightarrow \operatorname{Tor}_1^R(H^{q+1}(G, RG), L) \longrightarrow 0$$

We use this Lemma in the case $R = \mathbb{Z}$ (hereditary since $\mathbb{Z}$ is a PID [Rot09, 4.12]), $L = \mathbb{F}_3$ and $q = 2$, G is FP_∞ since it acts properly and cocompactly on $\mathcal{U}(W, K)$. A calculation in [Dav08, Example 8.5.8] gives

$$H^3(G, \mathbb{Z}G) = H^3(W, \mathbb{Z}W) = \mathbb{F}_2$$

$$H^2(G, \mathbb{Z}G) = H^2(W, \mathbb{Z}W) = \mathbb{Z}^\infty$$

$\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{F}_2, \mathbb{F}_3) = 0$ so $H^2(G, \mathbb{F}_2G) = H^2(W, \mathbb{F}_3W) = \mathbb{F}_3^\infty$ and $\mathrm{cd}_{\mathbb{F}_2} = 2$.

Ian Leary has pointed out to the author that much of the following argument appears in [DL98, proof of Theorem 2].

Example 2.29. A (not torsion-free) group with $\mathcal{O}_\mathcal{F} \mathrm{cd}_{\mathbb{F}_3} G = 2$ and $\mathcal{O}_\mathcal{F} \mathrm{cd}_{\mathbb{Z}} G = 3$.

Consider the group W of the above example (don't pass to a finite index torsion-free subgroup). $\mathcal{U}(W, K)$ is known to be a model for $\mathrm{E}_{\mathcal{F}in} W$ [Dav08, Theorem 12.3.4(ii)] and thus $\mathcal{O}_\mathcal{F} \mathrm{cd}_{\mathbb{Z}} W \leq 3$. To see that $\mathcal{O}_\mathcal{F} \mathrm{cd}_{\mathbb{Z}} W = 3$ calculate as in [LN03, p.147], where X^{sing} denotes the singular set of a CW complex X - the subcomplex with non-trivial isotropy.

$$\begin{aligned} \mathrm{Ext}^3_{\mathcal{O}_\mathcal{F}}(\mathbb{Z}(-), \mathbb{Z}[-, G/1]) &\cong H^3_G(\mathcal{U}(W, K), \mathcal{U}(W, K)^{\mathrm{sing}}; \mathbb{Z}G) \\ &\cong H^3_G(\mathcal{U}(W, K), \mathcal{U}(W, \partial K); \mathbb{Z}G) \qquad (\star) \\ &\cong H^3 \operatorname{Hom}_{\mathbb{Z}G}(C_*(\mathcal{U}(W, K), \mathcal{U}(W, \partial K)), \mathbb{Z}G) \end{aligned}$$

Recall $\mathcal{U}(W, K) = W \times K / \sim$ where the identification is only on $W \times \partial K$ and $(K, \partial K) \simeq (\mathcal{C}\mathbb{RP}^2, \mathbb{RP}^2)$. Here $\mathcal{C}X$ denotes the cone on a space X. $\mathcal{U}(W, \partial K)$ is precisely the subset of $\mathcal{U}(W, K)$ with non-trivial isotropy. The cochain complex

$$K^* = \operatorname{Hom}_{\mathbb{Z}G} \left(C_*(\mathcal{U}(W, K), \mathcal{U}(W, \partial K)), \mathbb{Z}G \right)$$

is generated by $\mathbb{Z}G$-maps $f : C_n(\mathcal{U}(W, K)) \to \mathbb{Z}G$ vanishing on $\mathcal{U}(W, \partial K)$. Fix some $K_0 \subset \mathcal{U}(W, K)$, a copy of K inside $\mathcal{U}(W, K)$. A map f, non-zero on only one G-orbit of cells in $C_*(\mathcal{U}(W, K), \mathcal{U}(W, \partial K))$, is completely determined by the value it takes on $C_n(K_0) \cong C_n(K)$ and an element $g \in G$. K^* is generated by such maps so we conlude $K^* \cong C^*(K, \partial K) \otimes_{\mathbb{Z}} \mathbb{Z}G \cong C^*(\mathcal{C}\mathbb{RP}^2, \mathbb{RP}^2) \otimes_{\mathbb{Z}} \mathbb{Z}G$.

$$\begin{aligned} H^3_G(\mathcal{U}(W, K), \mathcal{U}(W, \partial K); \mathbb{Z}G) &\cong H^3(\mathcal{C}\mathbb{RP}^2, \mathbb{RP}^2; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}G \\ &\cong H^2(\mathbb{RP}^2; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}G \\ &= \mathbb{F}_2 G \end{aligned}$$

Where the last isomorphism is from the long exact sequence of the pair $(\mathcal{C}\mathbb{RP}^2, \mathbb{RP}^2)$. Now $(\star)$ implies $\mathcal{O}_\mathcal{F} \mathrm{cd}_{\mathbb{Z}} W \geq 3$ and so in fact $\mathcal{O}_\mathcal{F} \mathrm{cd}_{\mathbb{Z}} W = 3$.

$\mathcal{U}(W, \partial K)$ is the singular set of $\mathcal{U}(W, K)$, so in particular the fixed point sets of finite subgroups (except for the trivial subgroup) agree. They are contractible and hence $\mathbb{F}_3$-acyclic. Since $\mathcal{U}(W, \partial K)$ is also $\mathbb{F}_3$-acyclic, taking the Bredon chain complex

$$P_* : G/H \mapsto C_*(G/H) \otimes_{\mathbb{Z}} \mathbb{F}_3$$

where $C_*(-)$ is the usual Bredon chain complex associated to $\mathcal{U}(W, \partial K)$ gives $\mathcal{O}_\mathcal{F} \mathrm{cd}_{\mathbb{F}_3} W \leq 2$.

W is a right-angled Coxeter group so its spherical subgroups correspond to simplices in L, the finite subgroup corresponding to an n-simplex is $\oplus_1^n \mathbb{Z}_2$ and has order 2^n. Any finite subgroup is subconjugated to a spherical subgroup [Dav08, Theorem 12.3.4(i)], so any finite subgroup of W has order a power of 2, thus W has no $\mathbb{F}_3$-torsion. By Corollary 2.26, $\mathcal{O}_{\mathcal{F}} \operatorname{cd}_{\mathbb{F}_3} W = 1$ if and only if $\mathcal{O}_{\mathcal{F}} \operatorname{cd}_{\mathbb{Z}} W = 1$ but we have already shown that $\mathcal{O}_{\mathcal{F}} \operatorname{cd}_{\mathbb{Z}} W = 3$ proving that $\mathcal{O}_{\mathcal{F}} \operatorname{cd}_{\mathbb{F}_3} W \neq 1$ and in fact $\mathcal{O}_{\mathcal{F}} \operatorname{cd}_{\mathbb{F}_3} W = 2$.

Example 2.30. A Group with $\operatorname{cd}_{\mathbb{Q}} G \neq \mathcal{O}_{\mathcal{F}} \operatorname{cd}_{\mathbb{Q}} G$

In [LN03], Leary and Nucinkis construct examples of virtually torsion-free groups with $\operatorname{vcd}_{\mathbb{Z}} G = nm$ and $\mathcal{O}_{\mathcal{F}} \operatorname{cd}_{\mathbb{Q}} G = m(n+1)$ for various integers n and m. The construction relies on [LN03, Theorem 6], we show that groups constructed using this theorem have $\mathcal{O}_{\mathcal{F}} \operatorname{cd}_{\mathbb{Q}} G = m(n+1)$ as well. So since $\operatorname{cd}_{\mathbb{Q}} G \leq \operatorname{vcd}_{\mathbb{Z}} G$ this provides examples of groups with $\operatorname{cd}_{\mathbb{Q}} G \neq \mathcal{O}_{\mathcal{F}} \operatorname{cd}_{\mathbb{Q}} G$.

All that is needed is to prove that groups G satisfying the assumptions of [LN03, Theorem 6] satisfy $\mathcal{O}_{\mathcal{F}} \operatorname{cd}_{\mathbb{Q}} G \geq m(n+1)$ also, since combining this with the inequality $\mathcal{O}_{\mathcal{F}} \operatorname{cd}_{\mathbb{Q}} G \leq \mathcal{O}_{\mathcal{F}} \operatorname{cd}_{\mathbb{Z}} G$ will give $\mathcal{O}_{\mathcal{F}} \operatorname{cd}_{\mathbb{Q}} G = m(n+1)$ as required. As part of Leary and Nucinkis' proof, they show that for a model X for $\mathrm{E}_{\mathfrak{Fin}} G$, the cellular chain complex $C_*(X^{m(n+1)}, (X^{m(n+1)})^{\text{sing}})$ contains a copy of $\mathbb{Z}G$ in dimension $m(n+1)$ as a direct summand. Here X^i denotes the i skeleton of some CW complex X and X^{sing} is the singular subcomplex of X - those cells of X having non-trivial isotropy. Using Lemma 2.32 below,

$$\begin{aligned} H^{m(n+1)}_{\mathcal{O}_{\mathcal{F}}}(G, \mathbb{Q}[-, G/1]) &\cong H^{m(n+1)}_G(C_*(X, X^{\text{sing}}); \mathbb{Q}G) \\ &\cong H^{m(n+1)}_G(C_*(X^{m(n+1)}, (X^{m(n+1)})^{\text{sing}}); \mathbb{Q}G) \\ &\neq 0 \end{aligned}$$

Showing $\mathcal{O}_{\mathcal{F}} \operatorname{cd}_{\mathbb{Q}} G \geq m(n+1)$.

The examples constructed with this method can never be of type $\mathcal{O}_{\mathcal{F}} \mathrm{FP}_\infty$ [LN03, Question 2, p.154], so a natural question is whether this phenonemon can occur for groups of this type:

Question 2.31. Do there exist groups of type $\mathcal{O}_{\mathcal{F}} \mathrm{FP}_\infty$ with $\operatorname{cd}_{\mathbb{Q}} G \neq \mathcal{O}_{\mathcal{F}} \operatorname{cd}_{\mathbb{Q}} G$?

Lemma 2.32 For any group G and model X for $\mathrm{E}_{\mathfrak{Fin}} G$,

$$H^*(G, R[-, G/1]) \cong H^*_G(C_*(X, X^{\text{sing}}); RG)$$

Where $C_*(X, X^{\text{sing}})$ denotes the cellular chain complex of RG-modules associated to the pair (X, X^{sing}).

Proof. Firstly, if $C_*(X^-)$ denotes the cellular chain complex of X as a contravariant $\mathcal{O}_{\mathcal{F}}$-module,

$$H^*\left(\operatorname{Mor}_{\mathcal{O}_{\mathcal{F}}}\left(C_*(X^{\text{sing}})^{-}, R[-, G/1]\right)\right) = 0$$

Since the G-orbits of cells in X^{sing} all give rise to contravariant modules of the form $R[-, G/H]$ for $H \neq 1$, and by the Yoneda-type Lemma 1.5,

$$\operatorname{Mor}_{\mathcal{O}_{\mathcal{F}}}(R[-, G/H], R[-, G/1]) \cong R[G/H, G/1] = 0$$

Using the long exact sequence in homology associated to the pair (X, X^{sing}),

$$\begin{aligned} H^*_{\mathcal{O}_\mathcal{F}}(G, R[-, G/1]) &\cong H^* \operatorname{Mor}_{\mathcal{O}_\mathcal{F}}(C_*(X^-), R[-, G/1]) \\ &\cong H^* \operatorname{Mor}(C_*(X^-, (X^{\text{sing}})^-), R[-, G/1]) \qquad (\star) \end{aligned}$$

Via the Yoneda-type Lemma 1.5, there is a chain of natural isomorphisms

$$\begin{aligned} &\operatorname{Mor}_{\mathcal{O}_\mathcal{F}}(C_*(X^-, (X^{\text{sing}})^-), R[-, G/1]) \\ &\cong \operatorname{Mor}_{\mathcal{O}_\mathcal{F}}\left(\bigoplus_{\substack{G\text{-orbits of } i\text{-cells} \\ \text{with trivial isotropy}}} R[-, G/1], R[-, G/1] \right) \\ &\cong \prod \operatorname{Mor}_{\mathcal{O}_\mathcal{F}}(R[-, G/1], R[-, G/1]) \\ &\cong \prod \operatorname{Hom}_{RG}(RG, RG) \\ &\cong \operatorname{Hom}_{RG}\left(\bigoplus RG, RG\right) \\ &\cong \operatorname{Hom}_{RG}(C_*(X, X^{\text{sing}}), RG) \end{aligned}$$

Thus

$$H^* \operatorname{Mor}_{\mathcal{O}_\mathcal{F}}(C_*(X^-, (X^{\text{sing}})^-) \cong H^* \operatorname{Hom}_{RG}(C_*(X, X^{\text{sing}}), RG)$$

and combining this with the isomorphism $(\star)$ completes the proof. □

2.7 Contravariant FP_n Conditions

This subsection builds up to Corollary 2.35, that a group G is $\mathcal{O}_\mathcal{F}\,\mathrm{FP}_0$ over R if and only if it has finitely many conjugacy classes of finite subgroups and is $\mathcal{O}_\mathcal{F}\,\mathrm{FP}_n$ over R if and only if it is $\mathcal{O}_\mathcal{F}\,\mathrm{FP}_0$ and the Weyl groups $WK = N_G K/K$ are FP_n over R for all finite subgroups K. Again, this result is well known when $R = \mathbb{Z}$ but hasn't been written down for general rings, although none of the proofs require any substantial alteration to do this.

Proposition 2.33 [KMPN09, Lemma 3.1] G is $\mathcal{O}_\mathcal{F}\,\mathrm{FP}_0$ over R if and only if G has finitely many conjugacy classes of finite subgroups.

Notice that this is independent of the ring R, so when speaking of $\mathcal{O}_\mathcal{F}\,\mathrm{FP}_0$ we needn't mention the ring R.

Proof. If G is $\mathcal{O}_\mathcal{F}\,\mathrm{FP}_0$ then there is a finitely generated free contravariant module F and an epimorphism $F \twoheadrightarrow \underline{R}$, since F is free there is a G-finite G-set Ω with finite stabilisers such that $F = R[-, \Omega]$. Let G_x denote the point stabiliser of $x \in \Omega$, since $gG_xg^{-1} = G_{gx}$ for any $g \in G$, there is at most one conjugacy class for each orbit. There are only finitely many orbits so we may deduce there is only a finite set of conjugacy classes of finite subgroups of point stabilisers of Ω.

Let K be a finite subgroup of G, evaluating $R[-, \Omega]$ at G/K gives a surjection

$$R[G/K, \Omega] = R[\Omega^K] \twoheadrightarrow R$$

This implies that Ω^K is non-empty, so K stabilises a point and is a subgroup of a point stabiliser and hence a member of one of the finite set of conjugacy classes of finite subgroups of point stabilisers.

For the converse, if G has only finitely many conjugacy classes of finite subgroups then we may take $\Omega = \coprod_{H \in X} G/H$ where H runs over the set of conjugacy class representatives X of finite subgroups of G. Now if $K \leq G$ is a finite subgroup

$$R[K, \Omega] = R[\Omega^K] = \bigoplus_{H \in X} R[(G/H)^K]$$

But $K = gHg^{-1}$ for some $H \in X$ and $g \in G$ so $gH \in (G/H)^K$ so the augmentation map $R[-, \Omega] \longrightarrow \underline{R}$ is a surjection when evaluated at any G/K and hence is an epimorphism of contravariant modules. □

Proposition 2.34 [KMPN09, Lemma 3.2] Let G be $\mathcal{O}_\mathcal{F}\,\mathrm{FP}_0$, then a contravariant module $M(-)$ is $\mathcal{O}_\mathcal{F}\,\mathrm{FP}_n$ $(n \geq 1)$ over R if and only if $M(G/K)$ is of type FP_n over $R[WK]$ for all finite subgroups $K \leq G$.

Proof. Let M be a contravariant module of type $\mathcal{O}_\mathcal{F}\,\mathrm{FP}_n$ and $P_* \twoheadrightarrow M$ a projective resolution, by a Bredon cohomology analogue of [Bro94, VIII4.3,4.5] we may assume that all P_i for $i \leq n$ are finitely generated free Bredon modules. Evaluating this resolution at G/H for a finite subgroup H, and applying Corollary 2.7, we deduce each $P_i(G/H)$ is a finite direct product of projective FP_∞ WH-modules and hence finitely generated. So we have constructed a projective resolution of $M(G/K)$ which is finitely generated up to degree n.

For the converse we use induction on n. Let $n = 0$ and M a contravariant module with $M(G/K)$ of type FP_0, ie. finitely generated, over $R[WK]$. We construct a finitely generated free module F with an epimorphism $F \twoheadrightarrow M$, thus showing that M is finitely generated and hence $\mathcal{O}_\mathcal{F}\,\mathrm{FP}_0$ over R.

If $H \in X$ and $K = gHg^{-1}$ then the map $K \mapsto gH$ induces a G-bijection between G/H and G/K with inverse $H \mapsto g^{-1}H$. Hence $M(G/H)$ and $M(G/K)$ are isomorphic via the maps $M(K \mapsto gH)$ and $M(H \mapsto gK)$. Similarly $R[G/K, G/H]$ and $R[G/H, G/H]$ are isomorphic via the maps $R[K \mapsto gH, G/H]$ and $R[H \mapsto g^{-1}K, G/H]$. By assumption $M(G/H)$ is finitely generated, say with a generating set of size n, choose a morphism

$$\bigoplus_1^n R[-, G/H] \longrightarrow M(-)$$

which is an epimorphism when evaluated at G/H, such a morphism can always be chosen by a Yoneda-type Lemma argument [MV03, p.9], which also tells us that we have the following commutative diagram

$$\begin{array}{ccc}
\bigoplus_1^n R[G/H, G/H] & \longrightarrow & M(G/H) \\
\downarrow & & \downarrow \\
\bigoplus_1^n R[G/K, G/H] & \longrightarrow & M(G/K)
\end{array}$$

where the left and right maps are bijections and the top map is an epimorphism, thus the bottom map is also an epimorphism. Hence the map

$$\bigoplus_{1}^{n} R[-, G/H] \longrightarrow M(-)$$

is an epimorphism when evaluated at any conjugate of H. Taking the direct sum of these:

$$\bigoplus_{H \in \mathcal{F}in/G} R[-, G/H] \longrightarrow M(-)$$

Where $\mathcal{F}in/G$ denotes the set of conjugacy classes of finite subgroups, provides a finitely generated free module with an epimorphism onto $M(-)$.

Now suppose $n > 0$ and the claim is true for all $k < n$. $M(G/K)$ is a $R[WK]$-module of type FP_n over R, so in particular it is FP_0 over R and finitely generated. Let $K_0 \hookrightarrow P_0 \twoheadrightarrow M$ be a short exact sequence in contravariant modules with P_0 finitely generated free. By the argument of the first paragraph, for any finite subgroup H, $P_0(G/H)$ is a $R[WH]$-module of type FP_∞ over R and by [Bie81, Proposition 1.4] $K_0(G/H)$ is FP_{n-1} over R and by induction, K_0 is $\mathcal{O}_\mathcal{F}\,\mathrm{FP}_{n-1}$ over R. □

Corollary 2.35 The following are equivalent for a group G

1. G is $\mathcal{O}_\mathcal{F}\,\mathrm{FP}_n$ over R.
2. G is $\mathcal{O}_\mathcal{F}\,\mathrm{FP}_0$ and the Weyl groups WK are FP_n over R for all finite subgroups K.
3. G is $\mathcal{O}_\mathcal{F}\,\mathrm{FP}_0$ and the centralisers C_GK are FP_n over R for all finite subgroups K.

Proof. By the previous Proposition (1) and (2) are equivalent. To see the equivalence of (2) and (3) consider the short exact sequence

$$0 \longrightarrow K \longrightarrow N_GK \longrightarrow WK \longrightarrow 0$$

K is finite and hence FP_∞, so WK is FP_n over R if and only if N_GK is FP_n over R. [Bie81, Proposition 2.7] K is finite, so C_GK is finite index in N_GK [Rob96, 1.6.13] and so C_GK is FP_n over R if and only if N_GK is FP_n over R. Combining the last two results gives WK is FP_n over R if and only if C_GK is FP_n over R. □

In view of [Bie81, Proposition 2.1], that G is FP_1 over R if and only if its finitely generated we have the following.

Corollary 2.36 G is $\mathcal{O}_\mathcal{F}\,\mathrm{FP}_1$ over R if and only if it has finitely many conjugacy classes of finite subgroups and all the Weyl groups of finite subgroups are finitely generated.

Example 2.37. In [BS80], it's shown that Abels' group is FP_2 over $\mathbb{Q}$ but not over $\mathbb{Z}$. The Bestvina Brady groups also provide examples of groups which are FP_n over some rings but not others [BB97]. By taking finite index extensions, groups can be produced with the same property but that are not $\mathcal{O}_\mathcal{F}\,\mathrm{FP}_0$ and groups that are $\mathcal{O}_\mathcal{F}\,\mathrm{FP}_n$ over some rings but not over others [LN03].

2.8 Finitely Generated Projectives and Duality

This section grew out of an investigation into which groups were $\mathcal{O}_\mathcal{F}$ FP over some ring R with

$$H^i_{\mathcal{O}_\mathcal{F}}(G, R[\not{-}, ?]) \cong \begin{cases} \underline{R}(?) & \text{if } i = n \\ 0 & \text{else.} \end{cases}$$

We prove in Theorem 2.46 that the only groups satisfying this property are torsion-free, and hence torsion-free Poincaré duality groups over R. A number of technical results concerning duality of Bredon modules are needed to show this, they are all analogs of results for modules over group rings that can be found in [Bie81].

For $M(-)$ a contravariant module, denote by $M(-)^D$ the dual module

$$M(?)^D = \mathrm{Mor}_{\mathcal{O}_\mathcal{F}}(M(\not{-}), R[\not{-}, ?])$$

Similarly for $A(-)$ a covariant module:

$$A(?)^D = \mathrm{Mor}_{\mathcal{O}_\mathcal{F}}(A(\not{-}), R[?, \not{-}])$$

Example 2.38. If G is an infinite group and $\underline{R}(-)$ is the covariant constant functor on R then $\underline{R}(-)^D = 0$,

$$\begin{aligned} \underline{R}(-)^D &= \mathrm{Mor}_{\mathcal{O}_\mathcal{F}}(\underline{R}(\not{?}), R[-, \not{?}]) \\ &\cong \mathrm{Mor}_{\mathcal{O}_\mathcal{F}}(\mathrm{Ind}_1^{\mathcal{O}_\mathcal{F}^{\mathrm{cov}}} R(\not{?}), R[-, \not{?}]) \\ &\cong \mathrm{Hom}_{RG}(R, R[-, G/1]) \end{aligned}$$

Using Example 2.9 and the adjointness of induction and restriction. Finally, $\mathrm{Hom}_{RG}(R, R[-, G/1])$ is the zero module since G is infinite.

Lemma 2.39 The dual functor takes projectives to projectives, and the double-dual functor $-^{DD} : \{\mathcal{O}_\mathcal{F}\text{-modules}\} \to \{\mathcal{O}_\mathcal{F}\text{-modules}\}$ is a natural isomorphism when restricted to the subcategory of finitely generated projectives.

Proof. By the Yoneda-type Lemma 1.5,

$$R[-, G/H]^D \cong \mathrm{Mor}_{\mathcal{O}_\mathcal{F}}(R[\not{?}, G/H], R[\not{?}, -]) \cong R[G/H, -]$$

The proof for covariant frees is identical.

For any module $M(-)$, there is a map $\zeta : M(-) \longrightarrow M(-)^{DD}$, given by $\zeta(m)(f) = f(m)$. If $M = R[-, G/H]$ then applying the Yoneda-type lemma twice shows $M(-)^{DD} = M(-)$. The duality functor represents direct sums, showing the double dual of a projective is also a projective.

Naturality follows from naturality of the map ζ. $\square$

2.8.1 Technical Results

We construct an R-module homomorphism

$$\nu : N(\not{?}) \otimes_{\mathcal{O}_\mathcal{F}} M(\not{?})^D \longrightarrow \mathrm{Mor}_{\mathcal{O}_\mathcal{F}}(M(\not{-}), N(\not{-}))$$

The main result of this section will be Lemma 2.41, that ν is an isomorphism when M is finitely generated projective and Proposition 2.45, that ν induces an isomorphism

$$H^i_{\mathcal{O}_\mathcal{F}}\left(G, R[-,?]\right) \otimes_{\mathcal{O}_\mathcal{F}} N(?) \cong H^i_{\mathcal{O}_\mathcal{F}}(G, N(?))$$

for all $i \leq n$ when G is $\mathcal{O}_\mathcal{F}\,\mathrm{FP}_n$.

Recall that elements of $N(?) \otimes_{\mathcal{O}_\mathcal{F}} M(?)^D$ are equivalence classes of elements

$$n_H \otimes \varphi_H \in \bigoplus_{G/H \in \mathcal{O}_\mathcal{F}} N(G/H) \otimes_R \mathrm{Mor}_{\mathcal{O}_\mathcal{F}}\left(M(-), R[-, G/H]\right)$$

For any $G/L \in \mathcal{O}_\mathcal{F}$ and $m \in M(G/L)$ we define

$$\begin{aligned} \nu\left(n_H \otimes_R \varphi_H\right)(G/L) : M(G/L) &\longrightarrow N(G/L) \\ m &\longmapsto N\left(\varphi_H(G/L)(m)\right)(n_H) \end{aligned}$$

This makes sense because $\varphi_H(G/L)(m) \in R[G/L, G/H]$ and N is a contravariant module so

$$N\big(\varphi_H(G/L)(m)\big) : N(G/H) \longrightarrow N(G/L)$$

We must check that $\nu(n_H \otimes_R \varphi_H)$ is a natural transformation, it's well defined including that it doesn't depend on the choice of equivalence class in $N(?) \otimes_{\mathcal{O}_\mathcal{F}} \mathrm{Mor}_{\mathcal{O}_\mathcal{F}}\big(M(-), R[-,?]\big)$, and that it is an R-module homomorphism.

$\nu(n_H \otimes_R \varphi_H)$ is a natural transformation:

Let $\alpha : G/L_1 \mapsto G/L_2$ be a G-map and $G/L_i \in \mathcal{O}_\mathcal{F}$. We must check the following diagram commutes:

$$\begin{array}{ccc} M(G/L_1) & \xrightarrow{\nu(n_H \otimes_R \varphi_H)(G/L_1) : m \mapsto N\left(\varphi_H(G/L_1)(m)\right)(n_H)} & N(G/L_1) \\ {\scriptstyle M(\alpha)}\uparrow & & \uparrow{\scriptstyle N(\alpha)} \\ M(G/L_2) & \xrightarrow{\nu(n_H \otimes_R \varphi_H)(G/L_2) : m \mapsto N\left(\varphi_H(G/L_2)(m)\right)(n_H)} & N(G/L_2) \end{array}$$

$$\begin{aligned} N(\alpha) \circ \big(\nu(n_H \otimes_R \varphi_H)(G/L_2)\big)(m) &= N(\alpha) \circ N\left(\varphi_H(G/L_2)(m)\right)(n_H) \\ &= N\big(\varphi_H(G/L_2)(m) \circ \alpha\big)(n_H) \\ &= N\big(\left(R[\alpha, G/H] \circ \varphi_H(G/L_2)\right)(m)\big)(n_H) \\ &= N\big((\varphi_H(G/L_1) \circ M(\alpha))(m)\big)(n_H) \\ &= \big(\nu(n_H \otimes_R \varphi_H)(G/L_2) \circ M(\alpha)\big)(m) \end{aligned}$$

Where the second equality is because N is a contravariant functor, the third is because by definition $\varphi_H(G/L_2)(m) \circ \alpha = \left(R[\alpha, G/H] \circ \varphi_H(G/L_2)\right)(m)$, and the fourth is because φ_H is itself a natural transformation and hence following diagram commutes:

$$\begin{array}{ccc} M(G/L_1) & \xrightarrow{\varphi_H(G/L_1)} & R[G/L_1, G/H] \\ {\scriptstyle M(\alpha)}\uparrow & & \uparrow{\scriptstyle R[\alpha, G/H]} \\ M(G/L_2) & \xrightarrow{\varphi_H(G/L_2)} & R[G/L_2, G/H] \end{array} \qquad (\dagger)$$

ν is well-defined: Firstly,

$$\nu(rn_H \otimes \varphi_H) = \nu(n_H \otimes r\varphi_H)$$

This is because

$$\begin{aligned}\nu\left(n_H \cdot r \otimes_R \varphi_H\right)(G/L)(m) &= N\left(\varphi_H(G/L)(m)\right)(rn_H)\\ &= rN\left(\varphi_H(G/L)(m)\right)(n_H)\\ &= N\left(r\varphi_H(G/L)(m)\right)(n_H)\\ &= \nu(n_H \otimes r\varphi_H)\end{aligned}$$

Secondly, ν doesn't depend on the choice of equivalence class in:

$$N(?) \otimes_{\mathcal{O}_{\mathcal{F}}} \mathrm{Mor}_{\mathcal{O}_{\mathcal{F}}}\big(M(-), R[-, ?]\big)$$

Choose $n_H \in N(G/H)$, $\varphi_M \in \mathrm{Mor}_{\mathcal{O}_{\mathcal{F}}}(M(-), R[-, G/M])$, $\alpha : G/H \to G/M$ a G-map and $G/H, G/M \in \mathcal{O}_{\mathcal{F}}$, we must show that

$$\nu\big(N(\alpha)(n_H) \otimes_R \varphi_M\big) = \nu\left(n_H \otimes_R \left(\mathrm{Mor}_{\mathcal{O}_{\mathcal{F}}}(M(-), R[-, \alpha])\right)(\varphi_M)\right)$$

Let $G/L \in \mathcal{O}_{\mathcal{F}}$,

$$\begin{aligned}\nu\big(N(\alpha)(n_H) \otimes_R \varphi_M\big)(G/L)(m) &= N\left(\varphi_H(G/L_1)(m)\right)\left(N(\alpha)(n_H)\right)\\ &= N\left(\alpha \circ \varphi_H(G/L)(m)\right)(n_H)\\ &= N\left(R[G/L, \alpha](\varphi_H(G/L_1)(m))\right)(n_H)\\ &= N\left(\mathrm{Mor}_{\mathcal{O}_{\mathcal{F}}}(M(-), R[-, \alpha])(\varphi_H)(G/L_1)(m)\right)(n_H)\\ &= \nu\big(n_H \otimes_R \mathrm{Mor}_{\mathcal{O}_{\mathcal{F}}}(M(-), R[-, \alpha])(\varphi_M)\big)(G/L)(m)\end{aligned}$$

ν is a map of R-modules: It's clear that ν is additive, and

$$\nu(rn_H \otimes \varphi_H) = r\nu(n_H \otimes \varphi_H)$$

since $N(-)$ being a module over R implies that $N(\varphi_H(G/L)(m))$ is an R-module homomorphism.

Lemma 2.40 ν is natural in $N(-)$ in $M(-)$.

Proof. We only prove naturality in $N(-)$, the proof for $M(-)$ is similar. Let $F(-)$ be morphism of contravariant modules $N(-) \to N'(-)$, we must show that the following diagram of R-modules commutes.

$$\begin{array}{ccc} N(?) \otimes_{\mathcal{O}_{\mathcal{F}}} \mathrm{Mor}_{\mathcal{O}_{\mathcal{F}}}(M(-), R[-, ?]) & \xrightarrow{\nu_N} & \mathrm{Mor}_{\mathcal{O}_{\mathcal{F}}}(M(-), N(-)) \\ \downarrow{\scriptstyle F(?) \otimes_{\mathcal{O}_{\mathcal{F}}} \mathrm{Mor}_{\mathcal{O}_{\mathcal{F}}}(M(-), R[-, ?])} & & \downarrow{\scriptstyle \mathrm{Mor}_{\mathcal{O}_{\mathcal{F}}}(M(-), F(-))} \\ N'(?) \otimes_{\mathcal{O}_{\mathcal{F}}} \mathrm{Mor}_{\mathcal{O}_{\mathcal{F}}}(M(-), R[-, ?]) & \xrightarrow{\nu_{N'}} & \mathrm{Mor}_{\mathcal{O}_{\mathcal{F}}}(M(-), N'(-)) \end{array}$$

Let $n_H \otimes \varphi_H \in N(G/H) \otimes_{\mathcal{O}_{\mathcal{F}}} \mathrm{Mor}_{\mathcal{O}_{\mathcal{F}}}(M(-), R[-, G/H])$ then moving about the top right of the diagram yields

$$\big(\mathrm{Mor}_{\mathcal{O}_{\mathcal{F}}}(M(-), F(-)) \circ \nu_N(n_H \otimes \varphi_H)\big)(G/L)(m)$$

$$= F(G/L) \circ N(\varphi_H(G/L)(m))(n_H)$$

and the bottom left yields

$$\begin{aligned}
&\big(\nu_{N'} \circ F(\not{?}) \otimes \mathrm{Mor}_{\mathcal{O}_\mathcal{F}}(M(\not{-}), R[\not{-},\not{?}])(n_H \otimes \varphi_H)\big)(G/L)(m) \\
&\quad = \nu_{N'}\big(F(G/H)(n_H) \otimes \varphi_H)\big)(G/L)(m) \\
&\quad = N'\big(\varphi_H(G/L)(m)\big)\big(F(G/H)(n_H)\big)
\end{aligned}$$

That these two are equivalent is because F is a natural transformation, so the diagram below commutes.

$$\begin{array}{ccc}
N(G/L) & \xrightarrow{F(G/L)} & N'(G/L) \\
{\scriptstyle N(\varphi_H(G/L)(m))}\uparrow & & \uparrow{\scriptstyle N'(\varphi_H(G/L)(m))} \\
N(G/H) & \xrightarrow{F(G/H)} & N'(G/H)
\end{array}$$

□

The next lemma is an $\mathcal{O}_\mathcal{F}$ module version of [Bie81, Proposition 3.1].

Lemma 2.41 If $M(-)$ is finitely generated projective then ν is an isomorphism.

Proof. Consider first the case $M(-) = R[-, G/H]$, then the map ν becomes

$$\nu : N(\not{?}) \otimes_{\mathcal{O}_\mathcal{F}} \mathrm{Mor}\big(R[\not{-}, G/H], R[\not{-},\not{?}]\big) \longrightarrow \mathrm{Mor}_{\mathcal{O}_\mathcal{F}}\left(R[\not{-}, G/H], N(\not{-})\right)$$

But, using Lemmas 1.5 and 1.12, the left hand side collapses to

$$\begin{aligned}
N(\not{?}) \otimes_{\mathcal{O}_\mathcal{F}} \mathrm{Mor}\big(R[\not{-}, G/H], R[\not{-},\not{?}]\big) &\cong N(\not{?}) \otimes_R R[G/H,\not{?}] \\
&\cong N(G/H) \qquad (\star)
\end{aligned}$$

Under these isomorphisms $n_H \in N(G/H)$ maps to $n_H \otimes \mathrm{id}_H \in N(\not{?}) \otimes_R R[G/H,\not{?}]$ and then to $n_H \otimes \varphi$ where φ is the unique natural transformation φ with $\varphi(G/H)(\mathrm{id}_H) = \mathrm{id}_H$.

The right hand side collapses to

$$\mathrm{Mor}_{\mathcal{O}_\mathcal{F}}\left(R[\not{-}, G/H], N(\not{-})\right) \cong N(G/H) \qquad (\dagger)$$

again by the Yoneda-type Lemma 1.5, where n_H maps to the unique natural transformation ψ with $\psi(G/H)(\mathrm{id}) = n_H$.

$$\nu(n_H \otimes \varphi)(G/H)(\mathrm{id}_H) = N(\varphi(G/H)(\mathrm{id}_H))(n_H) = N(\mathrm{id}_H)(n_H) = n_H$$

Precomposing ν with the isomorphism from ($\star$) and postcomposing with the isomorphism from ($\dagger$) gives the identity map $N(G/H) \to N(G/H)$ and hence ν is an isomorphism.

The case for finitely generated free modules follows as all the necessary functors commute with finite direct sums, and for projectives from naturality of ν proved in Lemma 2.40. □

The following result is an analog of [Bie81, 5.2(a,c)].

Lemma 2.42 1. If $M(-)$ is finitely presented and $N(-)$ is flat then ν is an isomorphism.

2. If $M(-)$ is finitely generated and $N(-)$ is projective then ν is an isomorphism.

Proof. 1. If $F_1(-) \longrightarrow F_0(-) \longrightarrow M(-) \longrightarrow 0$ is an exact sequence with $F_i(-)$ finitely generated free then then by the naturality of ν and flatness of $N(-)$ we have the following commutative diagram with exact rows (for brevity we write Mor for $\mathrm{Mor}_{\mathcal{O}_\mathcal{F}}$, $\otimes$ for $\otimes_{\mathcal{O}_\mathcal{F}}$, and $M^*(?)$ for $\mathrm{Mor}\,(M(\not -), R[\not -, ?])$).

$$\begin{array}{ccccccc}
0 \longrightarrow & N(\not ?) \otimes M^*(\not ?) & \longrightarrow & N(\not ?) \otimes F_0^*(\not ?) & \longrightarrow & N(\not ?) \otimes F_1^*(\not ?) \\
 & \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow & \mathrm{Mor}(M(\not -), N(\not -)) & \longrightarrow & \mathrm{Mor}(F_0(\not -), N(\not -)) & \longrightarrow & \mathrm{Mor}(F_1(\not -), N(\not -))
\end{array}$$

The right hand and middle vertical maps are isomorphisms by Lemma 2.41, the result follows from the 5-Lemma.

2. If $F(?)$ is free then by Lemma 1.12 there is an isomorphism

$$F(\not ?) \otimes_{\mathcal{O}_\mathcal{F}} \mathrm{Mor}(M(\not -), R[\not -, \not ?]) \cong \mathrm{Mor}(M(\not -), F(\not -))$$

Checking the definition of this isomorphism shows it's induced by ν. If $N(?)$ is projective and $i : N(?) \hookrightarrow F(?)$ is a split injection then by naturality of ν, the following diagram commutes:

$$\begin{array}{ccc}
N(\not ?) \otimes_{\mathcal{O}_\mathcal{F}} \mathrm{Mor}(M(\not -), R[\not -, \not ?]) & \longrightarrow & \mathrm{Mor}(M(\not -), N(\not -)) \\
\downarrow & & \downarrow \\
F(\not ?) \otimes_{\mathcal{O}_\mathcal{F}} \mathrm{Mor}(M(\not -), R[\not -, \not ?]) & \xrightarrow{\cong} & \mathrm{Mor}(M(\not -), F(\not -))
\end{array}$$

Since i is a split injection, the left hand map is an injection and top map must be an injection. Consider the commutative diagram in the proof of part 1, only $F_0(-)$ is known to be projective so the middle vertical map is an isomorphism. Since $N(-)$ is projective the left and right hand vertical maps are monomorphisms and the Four Lemma completes the proof, implying that the left hand vertical map is an isomorphism.

□

We need the following quick technical lemma.

Lemma 2.43 If $P_*(-)$ is any chain complex of contravariant modules and $N(-)$ is any contravariant module, the following morphism is both well defined and natural in $P_*(-)$ and $N(-)$.

$$\xi^i : N(\not ?) \otimes_{\mathcal{O}_\mathcal{F}} H^i P_*(\not ?)^D \to H^i\big(N(\not ?) \otimes_{\mathcal{O}_\mathcal{F}} P_*(\not ?)^D\big)$$
$$\xi^i : N(\not ?) \otimes_{\mathcal{O}_\mathcal{F}} H^i(\mathrm{Mor}(P_*(\not -), R[\not -, \not ?]) \to H^i\big(N(\not ?) \otimes_{\mathcal{O}_\mathcal{F}} \mathrm{Mor}(P_*(\not -), R[\not -, \not ?])\big)$$
$$n_H \otimes [\varphi_H] \mapsto [n_H \otimes \varphi_H]$$

Where $H^i P_*(?)^D : G/H \mapsto H^i P_*(G/H)^D$.

Proof. If φ_H is a cocycle, $n_H \otimes \varphi_H$ is also a cocycle and similarly if φ_H is a coboundary then $n_H \otimes \varphi_H$ is a coboundary.

If $\alpha : G/L \to G/H$ is a G-map then by definition $\alpha_*[\varphi_H] = [\alpha_* \varphi_H]$ and

$$\begin{aligned} \xi^i\big(\alpha^* n_H \otimes [\varphi_H] - n_H \otimes \alpha_*[\varphi_H]\big) &= \xi^i\big(\alpha^* n_H \otimes [\varphi_H] - n_H \otimes [\alpha_* \varphi_H]\big) \\ &= [\alpha^* n_H \otimes \varphi_H - n_H \otimes \alpha_* \varphi_H] \\ &= 0 \end{aligned}$$

Finally naturality follows because the the functors $H^i(-)$ and $\mathrm{Mor}_{\mathcal{O}_{\mathcal{F}}}(-,?)$ are natural, and so is the process of taking tensor products. □

Since ν is natural (Lemma 2.40), if $P_*(-)$ is a projective resolution of $\underline{R}(-)$ by contravariant modules then ν induces chain homomorphisms

$$N(\not?) \otimes_{\mathcal{O}_{\mathcal{F}}} P_*(\not?)^D \longrightarrow \mathrm{Mor}_{\mathcal{O}_{\mathcal{F}}}(P_*(\not-), N(\not-))$$

Which in turn induce maps on cohomology

$$H^i\big(N(\not?) \otimes_{\mathcal{O}_{\mathcal{F}}} P_*(\not?)^D\big) \longrightarrow H^i_{\mathcal{O}_{\mathcal{F}}}(G, N(\not-))$$

Precomposing this with ξ^i gives a map

$$\nu^i : N(\not?) \otimes_{\mathcal{O}_{\mathcal{F}}} H^i_{\mathcal{O}_{\mathcal{F}}}(G, R[\not-, \not?]) \longrightarrow H^i_{\mathcal{O}_{\mathcal{F}}}(G, N(\not-))$$

Proposition 2.44 If G is $\mathcal{O}_{\mathcal{F}}\,\mathrm{FP}_n$ over R and $N(-)$ is projective then ν^i is an isomorphism for all $i \leq n$.

Proof. Choose a projective resolution $P_*(-) \twoheadrightarrow \underline{R}(-)$, finitely generated up to dimension n and write $K_i(-)$ for the i^{th} syzygy of $P_*(-)$. Since $N(-)$ is projective it is also flat and we have the following commutative diagram with exact rows, where we omit the ${}_{\mathcal{O}_{\mathcal{F}}}$ on $\otimes$, Mor, and H^i; and also write $M^*(?)$ for $\mathrm{Mor}(M(\not-), R[\not-, ?])$.

$$\begin{array}{ccccccc} N(\not?) \otimes P^*_{i-1}(\not?) & \longrightarrow & N(\not?) \otimes K^*_{i-1}(\not?) & \longrightarrow & N(\not?) \otimes H^i(G, R[\not-, \not?]) & \to & 0 \\ \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \nu^i} & & \\ \mathrm{Mor}(P_{i-1}(\not-), N(\not-)) & \to & \mathrm{Mor}(K_{i-1}(\not-), N(\not-)) & \longrightarrow & H^i(G, N(\not-)) & \longrightarrow & 0 \end{array}$$

Since G is $\mathcal{O}_{\mathcal{F}}\,\mathrm{FP}_n$, $K_{i-1}(-)$ and $P_{i-1}(-)$ are finitely generated, thus by Lemma 2.42 the middle and left hand vertical maps are isomorphisms. The 5-Lemma completes the proof. □

The following result is an analog of [Bie81, 9.1].

Proposition 2.45 If G is $\mathcal{O}_{\mathcal{F}}\,\mathrm{FP}$ over R, with $\mathcal{O}_{\mathcal{F}}\,\mathrm{cd}_R\, G = n$, and $N(-)$ is any contravariant module then there is a natural isomorphism:

$$\nu^n : N(\not?) \otimes_{\mathcal{O}_{\mathcal{F}}} H^n_{\mathcal{O}_{\mathcal{F}}}\left(G, R[\not-, \not?]\right) \cong H^n_{\mathcal{O}_{\mathcal{F}}}(G, N(\not-))$$

Proof. Let

$$0 \longrightarrow K(-) \longrightarrow F(-) \longrightarrow N(-) \longrightarrow 0$$

be a short exact sequence of contravariant modules over R with F free. By the naturality of ν^n we have the following commutative diagram with exact rows, we omit the $\mathcal{O}_\mathcal{F}$ decorations on $\otimes$ and H^* for brevity.

$$\begin{array}{ccccccc}
K(\not?) \otimes H^n(G, R[-, \not?]) & \rightarrowtail & F(\not?) \otimes H^n(G, R[-, \not?]) & \rightarrowtail & N(\not?) \otimes H^n(G, R[-, \not?]) & \rightarrowtail & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
H^n(G, K(-)) & \longrightarrow & H^n(G, F(-)) & \longrightarrow & H^n(G, N(-)) & \longrightarrow & 0
\end{array}$$

The middle vertical map is an isomorphism by Proposition 2.44, thus by the Four Lemma, the right hand vertical map is an epimorphism. Since there are no restrictions on $N(-)$, we conclude that the left hand vertical map is an epimorphism and by the Five Lemma that the right hand map is an isomorphism. $\square$

2.8.2 The Wrong Notion of Duality

Theorem 2.46 If G is an arbitrary group $\mathcal{O}_\mathcal{F}\,\mathrm{FP}$ group with $\mathcal{O}_\mathcal{F}\,\mathrm{cd}_R\, G = n$ and

$$H^i_{\mathcal{O}_\mathcal{F}}(G, R[\not-, ?]) \cong \begin{cases} \underline{R}(?) & \text{if } i = n \\ 0 & \text{else.} \end{cases}$$

then G is torsion-free.

Proof. Choose a length n finite type contravariant resolution $P_*(-)$ of $\underline{R}(-)$, then by Lemma 2.39(1) and the assumption on $H^n_{\mathcal{O}_\mathcal{F}}(G, R[\not-, ?])$, $P_*^D(-)$ is a covariant resolution by finitely generated projectives of $\underline{R}(-)$:

$$0 \longrightarrow P_0^D(-) \xrightarrow{\partial_1^D} P_1^D(-) \xrightarrow{\partial_2^D} \cdots \xrightarrow{\partial_n^D} P_n^D(-) \longrightarrow H^n_{\mathcal{O}_\mathcal{F}}(G, R[\not-, ?]) \cong \underline{R}(?) \longrightarrow 0$$

In particular, G has $\mathcal{O}_\mathcal{F}^{\mathrm{cov}}\,\mathrm{cd}_R\, G \leq n$, so by Theorem 2.15, G is R-torsion-free and $\mathrm{cd}_R\, G \leq n$. Choose a length n finite type projective RG-resolution Q_* of R, by Proposition 2.13 and Example 2.9,

$$(\mathrm{Ind}_1^{\mathcal{O}_\mathcal{F}^{\mathrm{cov}}} Q_*)(-) \twoheadrightarrow \underline{R}(-)$$

is a projective covariant resolution.

By the comparison theorem [Wei94, 2.2.6], the two projective covariant resolutions of $\underline{R}(-)$ are chain homotopy equivalent. Any additive functor preserves chain homotopy equivalences (a chain homotopy equivalence is defined purely with addition and function composition, which are preserved), so applying the dual functor to both complexes gives a chain homotopy equivalence between

$$0 \longrightarrow \underline{R}(-)^D \cong 0 \longrightarrow (\mathrm{Ind}_1^{\mathcal{O}_\mathcal{F}^{\mathrm{cov}}} Q_0)(-)^D \longrightarrow \cdots \longrightarrow (\mathrm{Ind}_1^{\mathcal{O}_\mathcal{F}^{\mathrm{cov}}} Q_n)(-)^D$$

and

$$0 \longrightarrow \underline{R}(-)^D \cong 0 \longrightarrow P_n(-)^{DD} \longrightarrow P_{n-1}(-)^{DD} \longrightarrow \cdots \longrightarrow P_0(-)^{DD}$$

(That $\underline{R}(-)^D \cong 0$ is just Example 2.38). We know both complexes above are left exact, since $\mathrm{Mor}_{\mathcal{O}_\mathcal{F}}$ is. Lemma 2.39 gives the commutative diagram below.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_n(-)^{DD} & \longrightarrow \cdots \longrightarrow & P_1(-)^{DD} & \longrightarrow & P_0(-)^{DD} \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ 0 & \longrightarrow & P_n(-) & \longrightarrow \cdots \longrightarrow & P_1(-) & \longrightarrow & P_0(-) \end{array}$$

The lower complex, $P_*(-)$, satisfies $H_0(P_*(-)) \cong \underline{R}(-)$ and $H_i(P_*(-)) = 0$ for all $i \neq 0$. Thus the same is true for the top complex, and also the complex $\mathrm{Ind}_1^{\mathcal{O}_\mathcal{F}^{\mathrm{cov}}} Q_*(-)^D$, since this is homotopy equivalent to it. In particular, there is an epimorphism of contravariant modules,

$$\mathrm{Ind}_1^{\mathcal{O}_\mathcal{F}^{\mathrm{cov}}} Q_n(-)^D \twoheadrightarrow \underline{R}(-)$$

The left hand side simplifies, using the adjointness of induction and restriction:

$$\mathrm{Ind}_1^{\mathcal{O}_\mathcal{F}^{\mathrm{cov}}} Q_n(-)^D = \mathrm{Mor}_{\mathcal{O}_\mathcal{F}}\left(\mathrm{Ind}_1^{\mathcal{O}_\mathcal{F}^{\mathrm{cov}}} Q_n, R[?, \not -]\right) \cong \mathrm{Hom}_{RG}(Q, R[?, G/1])$$

Since $\mathrm{Hom}_{RG}(Q, R[?, G/1]) = 0$ if $H \neq 1$, this module cannot surject onto $\underline{R}(-)$ unless G is torsion-free. □

2.9 Questions

Collected here are questions related to Bredon modules from this section.

Question 2.18. Is there a nice characterisation of the condition covariant-$\mathcal{O}_\mathcal{F}\,\mathrm{FP}_n$ over R, for groups which are not R-torsion free?

Question 2.27. What does the condition $\mathcal{O}_\mathcal{F}\,\mathrm{cd}_R\, G = 1$ represent? Is it equivalent to $\mathcal{O}_\mathcal{F}\,\mathrm{cd}_\mathbb{Z}\, G = 1$?

Question 2.31. Do there exist groups of type $\mathcal{O}_\mathcal{F}\,\mathrm{FP}_\infty$ with $\mathrm{cd}_\mathbb{Q}\, G \neq \mathcal{O}_\mathcal{F}\,\mathrm{cd}_\mathbb{Q}\, G$?

3 Mackey Functors

Throughout this section $\mathcal{F}$ will denote the family of finite subgroups. The definition of Mackey functors for subfamilies of $\mathcal{F}$ is identical, but some changes are needed for larger families such as that of virtually cyclic subgroups, see for example [Deg13b, §6.2].

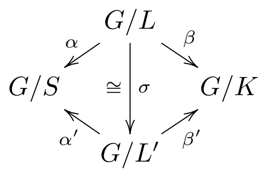
There are many constructions of Mackey functors, we use the construction coming from modules over a category, as we've already built up a lot of theory concerning these in Section 1. Two other constructions are mentioned in Remarks 3.9 and 3.10. We begin by building a small category $\mathcal{M}_{\mathcal{F}}$ then, using the language of Section 1, Mackey functors will be contravariant modules over $\mathcal{M}_{\mathcal{F}}$. Fix a commutative ring R. As in $\mathcal{O}_{\mathcal{F}}$, the objects of $\mathcal{M}_{\mathcal{F}}$ are the transitive G-sets with stabilisers in $\mathcal{F}$, the morphism set however is much larger. A *basic morphism* from G/H to G/K, where H and K are finite subgroups, is an equivalence class of diagrams of the form

$$G/H \xleftarrow{\alpha} G/L \xrightarrow{\beta} G/K$$

Where the maps are G-maps, and L is a finite subgroup of G. This basic morphism is equivalent to

$$G/H \xleftarrow{\alpha'} G/L' \xrightarrow{\beta'} G/K$$

if there is a bijective G-map $\sigma : G/L \to G/L'$, fitting into the commutative diagram below:



Form the free abelian monoid on these basic morphisms, and complete this free abelian monoid to a group, denoted $[G/H, G/K]_{\mathcal{M}_{\mathcal{F}}}$. This is the set of morphisms in $\mathcal{M}_{\mathcal{F}}$ from G/H to G/K.

Remark 3.1. When building the Mackey category, we could instead have started with equivalence classes of diagrams

$$G/H \leftarrow \Delta \to G/K$$

Where Δ is any finitely generated G-set with finite stabilisers and the maps are G-maps. This can be shown to be the free abelian monoid on the basic morphisms [TW95, Proposition 2.2]. Because of this alternative construction, we will pass freely between writing

$$(G/H \leftarrow G/L \to G/K) + (G/H \leftarrow G/L' \to G/K)$$

and

$$\Big(G/H \leftarrow G/L \coprod G/L' \to G/K\Big)$$

To complete the description of $\mathcal{M}_{\mathcal{F}}$, we must describe composition of morphisms. It's sufficient to describe composition of basic morphisms, and then use distributivity to extend this to all morphisms. If

$$G/H \leftarrow G/L \rightarrow G/K$$

and

$$G/K \leftarrow G/S \rightarrow G/Q$$

are two basic morphisms then their composition is the pullback of the diagram below in the category of G-sets

$$\begin{array}{ccccccc} & G/L & & & G/S & & \\ \swarrow & & \searrow & & \swarrow & & \searrow \\ G/H & & & G/K & & & G/Q \end{array}$$

Before we describe this pullback explicitly, some notation:

Remark 3.2. If H is a finite subgroup of G, we'll use the notation H^g to mean the conjugate $g^{-1}Hg$. Thus there is always a G-map

$$\begin{aligned} \alpha_g : G/H &\longrightarrow G/H^g \\ H &\mapsto g(g^{-1}Hg) \end{aligned}$$

and a G-map

$$\begin{aligned} \alpha_{g^{-1}} : G/H^g &\longrightarrow G/H \\ (g^{-1}Hg) &\mapsto g^{-1}H \end{aligned}$$

Lemma 3.3 [MPN06, §3] **Composition of morphisms in $\mathcal{M}_{\mathcal{F}}$.**
The diagram below is a pullback in the category of G-sets.

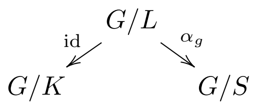
$$\sum_{x \in L^g \backslash K / S^{g'}} \left(\begin{array}{ccccc} & & G/\big(L^g \cap S^{g'x^{-1}}\big) & & \\ & \swarrow^{\alpha_{g^{-1}}} & & \searrow^{\alpha_{x(g')^{-1}}} & \\ G/L & & & & G/S \\ & \searrow_{\alpha_g} & & \swarrow_{\alpha_{g'}} & \\ & & G/K & & \end{array} \right)$$

Notice that the subgroup $L^g \cap S^{g'x^{-1}}$ is both a subgroup of K via the maps on the left and subconjugated to K via the map α_x, which is the composition of the maps on the right.

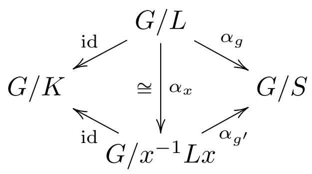
Remark 3.4. The pullback of Lemma 3.3 could be written as:

$$\sum_{x \in L^g \backslash K / S^{g'}} \left(\begin{array}{ccccc} & & G/\big(L^{gx^{-1}} \cap S^{g'}\big) & & \\ & \swarrow^{\alpha_{xg^{-1}}} & & \searrow^{\alpha_{(g')^{-1}}} & \\ G/L & & & & G/S \\ & \searrow_{\alpha_g} & & \swarrow_{\alpha_{g'}} & \\ & & G/K & & \end{array} \right)$$

Lemma 3.5 Standard form for morphisms in $\mathcal{M}_{\mathcal{F}}$. [TW95, Lemma 2.1] Any basic morphism is equivalent to one in the standard form:



Recall that two such basic morphisms are equivalent if there is a commutative diagram of the form:



The commutativity of the left hand triangle insures that $x \in K$, and that of the right hand diagram gives $\alpha_g = \alpha_{g'} \circ \alpha_x$, or more consisely $gS = xg'S$. This means $KgS = Kg'S$ and $x = gS(g')^{-1} \cap K = gSg^{-1} \cap K$. Thus a basic morphism is determined by both an element of $K \backslash G/S$ and a subgroup L, subconjugate to K, unique up to conjugation by an element $x \in gSg^{-1} \cap K$. In summary,

$$[G/K, G/S]_{\mathcal{M}_{\mathcal{F}}} = \bigoplus_{g \in K \backslash G/S} \bigoplus_{\substack{L \leq gSg^{-1} \cap K \\ \text{Up to } gSg^{-1} \cap K\text{-conjugacy}}} \mathbb{Z}_{L,g} \tag{1}$$

Example 3.6. If $S = 1$ then (1) becomes

$$[G/K, G/1]_{\mathcal{M}_{\mathcal{F}}} = \bigoplus_{g \in K \backslash G} \mathbb{Z}_g \cong \mathbb{Z}[K \backslash G]$$

Remark 3.7. The category $\mathcal{M}_{\mathcal{F}}$ has property (A) by construction, but it does not have property (EI) (See the beginning of Section 1 and Remark 1.1 for the definitions of these properties). For example, given any non-trivial finite subgroup H of G, the endomorphism

$$e = \left(G/H \xleftarrow{\alpha_1} G/1 \xrightarrow{\alpha_1} G/H \right)$$

is not an isomorphism. If

$$m = \left(G/H \xleftarrow{\alpha_1} G/K \xrightarrow{\alpha_g} G/H \right)$$

is some other basic morphism then their composition is

$$m \circ e = \sum_{x \in H/K} \left(G/H \xleftarrow{\alpha_1} G/1 \xrightarrow{\alpha_{xg}} G/H \right)$$

So it's clear that composing e with any element of $[G/H, G/H]_{\mathcal{H}_{\mathcal{F}}}$ can never produce the identity morphism on G/H. The structure of the endomorphisms and automorphisms of objects in $\mathcal{H}_{\mathcal{F}}$ is explained in Remarks 3.11 and 3.14.

As described in Section 1, a covariant (resp. contravariant) module over the Mackey category, or $\mathcal{M}_{\mathcal{F}}$-module, is a covariant functor (resp. contravariant functor) from $\mathcal{M}_{\mathcal{F}}$ to **R-Mod**. Following [MPN06], we will mostly consider contravariant Mackey functors. Indeed from here on, whenever we write $\mathcal{M}_{\mathcal{F}}$-module, we mean contravariant $\mathcal{M}_{\mathcal{F}}$-module. Following the notation of Section 1, $R[-, G/H]_{\mathcal{M}_{\mathcal{F}}}$ denotes the free $\mathcal{M}_{\mathcal{F}}$-module with

$$R[-, G/H]_{\mathcal{M}_{\mathcal{F}}}(G/K) = R[G/K, G/H]_{\mathcal{M}_{\mathcal{F}}} = R \otimes_{\mathbb{Z}} [G/K, G/H]_{\mathcal{M}_{\mathcal{F}}}$$

Remark 3.8. The category of contravariant $\mathcal{M}_{\mathcal{F}}$-modules is isomorphic to the category of covariant $\mathcal{M}_{\mathcal{F}}$-modules.

Let $\zeta : \mathcal{M}_{\mathcal{F}} \to \mathcal{M}_{\mathcal{F}}^{\mathrm{op}}$ denote the contravariant functor mapping G/H to itself and

$$\zeta : (G/H \leftarrow G/L \to G/K) \longrightarrow (G/K \leftarrow G/L \to G/H)$$

It should be clear that $\zeta \circ \zeta = \mathrm{id}_{\mathcal{M}_{\mathcal{F}}}$. Let ζ_* be the map

$$\begin{aligned} \zeta_* : \{\text{Covariant } \mathcal{M}_{\mathcal{F}}\text{-modules}\} &\longrightarrow \{\text{Contravariant } \mathcal{M}_{\mathcal{F}}\text{-modules}\} \\ M(-) &\longmapsto M \circ \zeta(-) \end{aligned}$$

Then ζ_* is an isomorphism of categories, hence necessarily additive and exact. One can check that $\zeta_* R[G/H, -]_{\mathcal{M}_{\mathcal{F}}} = R[-, G/H]_{\mathcal{M}_{\mathcal{F}}}$, so ζ_* preserves projectives also. Because of this, there is no point considering finiteness conditions for both covariant and contravariant $\mathcal{M}_{\mathcal{F}}$-modules, a covariant $\mathcal{M}_{\mathcal{F}}$-module $M(-)$ has cohomological dimension n if and only if the contravariant $\mathcal{M}_{\mathcal{F}}$-module $\zeta_* M(-)$ has cohomological dimension n and similarly for the FP_n conditions.

Remark 3.9. Green's alternative description of the Mackey category.

There is an alternative description of Mackey modules, due to Green [Gre71], which we include here in full because when we later study cohomological Mackey functors in Section 4, we will need some of the language.

Green defined a Mackey functor $M(-)$ as a mapping,

$$M(-) : \{G/H \;:\; H \text{ a finite subgroup of } G\} \to \textbf{R-Mod}$$

with morphisms for any finite subgroups $K \leq H$ of G,

$$\begin{aligned} M(I_K^H) &: M(G/K) \to M(G/H) \\ M(R_K^H) &: M(G/H) \to M(G/K) \\ M(c_g) &: M(G/H) \to M(G/H^{g^{-1}}) \end{aligned}$$

Called *induction*, *restriction* and *conjugation* respectively. Induction is sometimes also called transfer. In the literature, $M(I_K^H)$, $M(R_K^H)$ and $M(c_g)$ are often written as just I_K^H, R_K^H and c_g - omitting the M entirely. We choose to use different notation so that we can identify I_K^H, R_K^H and c_g with specific morphisms in $\mathcal{M}_{\mathcal{F}}$ (see the end of this remark).

This mapping $M(-)$ must satisfy the following axioms,

(0) $M(I_H^H)$, $M(R_H^H)$ and $M(c_h)$ are the identity morphism for all $h \in H$.

(1) $M(R^K_J)\circ M(R^H_K) = M(R^H_J)$, where $J \leq K \leq H$ are finite subgroups of G.

(2) $M(I^H_K)\circ M(I^K_J) = M(I^H_J)$, where $J \leq K \leq H$ are finite subgroups of G.

(3) $M(c_g)\circ M(c_h) = M(c_{gh})$ for all $g, h \in G$.

(4) $M(R^{H^{g^{-1}}}_{K^{g^{-1}}})\circ M(c_g) = M(c_g)\circ M(R^H_K)$, where $K \leq H$ are finite subgroups and $g \in G$.

(5) $M(I^{H^{g^{-1}}}_{K^{g^{-1}}})\circ M(c_g) = M(c_g)\circ M(I^H_K)$, where $K \leq H$ are finite subgroups and $g \in G$.

(6) $M(R^H_J)\circ M(I^H_K) = \sum_{x\in J\backslash H/K} M(I^J_{J\cap K^{x^{-1}}})\circ M(c_x)\circ M(R^K_{J^x\cap K})$, where $J, K \leq H$ are finite subgroups of G.

Axiom (6) is often called the Mackey axiom. Converting between this description and our previous description is done by rewriting induction, restriction and conjugation in terms of morphisms of $\mathcal{M}_{\mathcal{F}}$.

$$\begin{aligned} M(I^H_K) &\longleftrightarrow M\big(G/H \xleftarrow{\alpha_1} G/K \xrightarrow{\alpha_1} G/K\big)\\ M(R^H_K) &\longleftrightarrow M\big(G/H \xleftarrow{\alpha_1} G/H \xrightarrow{\alpha_1} G/K\big)\\ M(c_g) &\longleftrightarrow M\big(G/H^{g^{-1}} \xleftarrow{\alpha_1} G/H^{g^{-1}} \xrightarrow{\alpha_g} G/H\big)\end{aligned}$$

Because of the above, we make the following definitions

$$\begin{aligned} I^H_K &= \big(G/H \xleftarrow{\alpha_1} G/K \xrightarrow{\alpha_1} G/K\big)\\ R^H_K &= \big(G/K \xleftarrow{\alpha_1} G/K \xrightarrow{\alpha_1} G/H\big)\\ c_g &= \big(G/H^{g^{-1}} \xleftarrow{\alpha_1} G/H^{g^{-1}} \xrightarrow{\alpha_g} G/H\big)\end{aligned}$$

It is possible to write any morphism in $\mathcal{M}_{\mathcal{F}}$ as a composition of the three morphisms above.

One can check that Green's axioms all follow from the description of the composition of morphisms in $\mathcal{M}_{\mathcal{F}}$ as pullbacks (Lemma 3.3), and vice versa. Complete proofs of the equivalence of this definition with our previous one can be found in [TW95, §2].

Remark 3.10. Dress's alternative description of the Mackey category

There is another alternative description of Mackey functors, due to Dress [Dre73], which describes a Mackey functor as a pair of functors $M_*, M^* : \mathcal{O}_{\mathcal{F}} \to$ **R-Mod**, where M_* is covariant and M^* is contravariant. We won't describe this here as we don't require it, a full description including a proof of equivalence with the previous two definitions can be found in [TW95, §2].

3.1 Free Modules

In this section we describe the structure of $\mathrm{Aut}(G/H)$ and $\mathrm{End}(G/H)$, and discuss free modules in the category of $\mathcal{M}_{\mathcal{F}}$-modules.

Remark 3.11. Structure of $\mathrm{Aut}(G/H)$.

As mentioned in Remark 3.7, $\mathcal{M}_{\mathcal{F}}$ doesn't have property (EI) - $\mathrm{End}(G/H)$ is not equal to $\mathrm{Aut}(G/H)$. Using the standard form of Lemma 3.5, the automorphisms of an object are the diagrams of the form

$$a_g = \big(G/H \xleftarrow{\alpha_1} G/H \xrightarrow{\alpha_g} G/H\big)$$

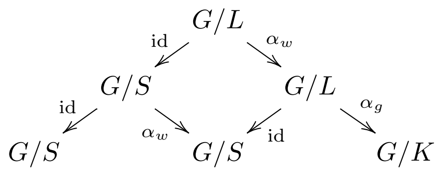
Where g is unique up to multiplication by an element of H. Every $g \in WH$ uniquely determines a G-map $\alpha_g : G/H \to G/H$ and every G-map comes from such a g. Finally, since $a_g \circ a_h = a_{hg}$, we determine that $\mathrm{Aut}(G/H) \cong \mathbb{Z}[WH^{\mathrm{op}}]$. This is identicial to the situation over the orbit category, where $\mathrm{Aut}_{\mathcal{O}_{\mathcal{F}}}(G/H) \cong \mathbb{Z}[WH^{\mathrm{op}}]$ also. Thus, as with $\mathcal{O}_{\mathcal{F}}$-modules, if $M(-)$ is a Mackey functor, then $M(G/H)$ is a right $R[WH^{\mathrm{op}}]$ module, equivalently a left $R[WH]$-module.

Lemma 3.12 As a left $R[W_GS]$ module, $R[G/S, G/K]_{\mathcal{M}_{\mathcal{F}}}$ is an $R[W_GS]$-permutation module with finite stabilisers. In addition, $R[G/1, G/K]_{\mathcal{M}_{\mathcal{F}}}$ is FP_∞ over RG.

Proof. The left action of $w \in W_GS$ on $[G/S, G/K]_{\mathcal{M}_{\mathcal{F}}}$ is the action given by pre-composing any basic morphism $G/S \overset{\mathrm{id}}{\leftarrow} G/L \overset{\alpha_g}{\rightarrow} G/K$ with the morphism $G/S \overset{\mathrm{id}}{\leftarrow} G/S \overset{\alpha_w}{\rightarrow} G/S$ to yield the morphism

$$G/S \overset{\alpha_1}{\leftarrow} G/L \overset{\alpha_{wg}}{\rightarrow} G/K$$

To show this we calculate the pullback



Under the identification (1), w maps $R_{L,g}$ onto $R_{L,wg}$, so the stabiliser of this action is the stabiliser of the action of $R[W_GS]$ on $R[S\backslash G/K]$, which is finite. In particular $R[G/S, G/K]_{\mathcal{M}_{\mathcal{F}}}$ is an $R[W_GS]$-permutation module with finite stabilisers. □

Remark 3.13. Unfortunately, $R[G/S, G/K]_{\mathcal{M}_{\mathcal{F}}}$ is not even W_GS finitely generated in general. For an example of this choose a group G with a finite subgroup K such that $K\backslash G$ has infinitely many W_GK-orbits (for a specific example see Example 2.11). Then, by Example 3.6,

$$R[G/K, G/1]_{\mathcal{M}_{\mathcal{F}}} \cong R[K\backslash G]$$

Which is not finitely generated as a left $R[W_GK]$ module.

Remark 3.14. Structure of $\mathrm{End}(G/H)$.

The structure of $\mathrm{End}(G/H)$ is more complex than that of $\mathrm{Aut}(G/H)$, a basic morphism in $\mathrm{End}(G/H)$ is determined by a morphism in standard form

$$e_{L,g} = \left(G/H \xleftarrow{\alpha_1} G/L \xrightarrow{\alpha_g} G/H\right)$$

where L is some subgroup of G. As such we can filter $\mathrm{End}(G/H)$ via the poset $\mathcal{F}/G$ of conjugacy classes of finite subgroups of G. If L is a finite subgroup of G then we write $\mathrm{End}(G/H)_L$ for the basic morphisms $e_{L,g}$ for all $g \in G$. Note that in particular, $\mathrm{End}(G/H)_H = \mathrm{Aut}(G/H)$. The abelian group $\mathrm{End}(G/H)_L$

is not closed under self-composition, but it is closed under pre-composition by elements of $\mathrm{Aut}(G/H)$, since

$$e_{L,g} \circ a_w \cong e_{L,wg}$$

Where $a_w = e_{H,w}$ as described in Remark 3.11. Thus $R\,\mathrm{End}(G/H)_L$ is a right $R\,\mathrm{Aut}(G/H)$ module, equivalently a left $R[WH]$ module. In summary, there is an $R[WH]$-module isomorphism

$$R\,\mathrm{End}(G/H) \cong \bigoplus_{L \in \mathcal{F}/G} R\,\mathrm{End}(G/H)_L$$

Where $\mathrm{End}(G/H)_H \cong \mathrm{Aut}(G/H)$.

Remark 3.15. $R\,\mathrm{End}(G/H)$ **is not in general** $R[WH]$ **finitely generated.**
Using (1), we see that

$$R\,\mathrm{End}(G/H)_1 \cong \bigoplus_{H\backslash G/H} R_{1,g}$$

With left action of $w \in W_G H$ taking $g \mapsto wg$. In other words, $R\,\mathrm{End}(G/H)_1 \cong R[H\backslash G/H]$ with the canonical action of $W_G H$. This is not in general finitely generated - take for example $G = D_\infty$, the infinite dihedral group generated by the involutions a and b, and $H = \langle a \rangle$. Then $W_G H$ is the trivial group but $H\backslash G/H$ is an infinite set so $R[H\backslash G/H]$ is not a finitely generated R-module.

3.2 Restriction, Induction and Coinduction

We specialise the constructions of Section 1.3 to the Mackey category. As well as the functors $\widehat{\mathrm{End}(G/H)} \longrightarrow \mathcal{M}_\mathcal{F}$, there are two useful functors $\mathcal{O}_\mathcal{F} \to \mathcal{M}_\mathcal{F}$, one covariant and the other contravariant: Let $\sigma : \mathcal{O}_\mathcal{F} \to \mathcal{M}_\mathcal{F}$ be the covariant functor sending

$$\sigma(G/H) = G/H$$

$$\sigma(G/H \overset{\alpha}{\to} G/K) = (G/H \overset{\mathrm{id}}{\leftarrow} G/H \overset{\alpha}{\to} G/K)$$

and $\tau : \mathcal{O}_\mathcal{F} \to \mathcal{M}_\mathcal{F}$ be the contravariant functor

$$\tau(G/H) = G/H$$

$$\tau(G/H \overset{\alpha}{\to} G/K) = (G/K \overset{\alpha}{\leftarrow} G/H \overset{\mathrm{id}}{\to} G/H)$$

Note that $\tau = \zeta \circ \sigma$. Thus σ induces restriction, induction, and coinduction between contravariant $\mathcal{O}_\mathcal{F}$-modules and (contravariant) $\mathcal{M}_\mathcal{F}$-modules and τ induces restriction, induction, and coinduction between covariant $\mathcal{O}_\mathcal{F}$-modules and (contravariant) $\mathcal{M}_\mathcal{F}$-modules.

Example 3.16. If $\underline{R}(-)$ is the constant covariant $\mathcal{O}_\mathcal{F}$-module then, recalling Example 2.9 that $\underline{R}(-) = \mathrm{Ind}_1^{\mathcal{O}_\mathcal{F}^{\mathrm{cov}}} R$,

$$\mathrm{Ind}_\tau \underline{R}(-) \cong \mathrm{Ind}_1^{\mathcal{M}_\mathcal{F}} R \cong R[-, G/1]_{\mathcal{M}_\mathcal{F}} \otimes_{RG} R$$

Since $R[G/K, G/1]_{\mathcal{M}_\mathcal{F}} \cong R[K\backslash G]$ by Example 3.6, this is the constant functor on R.

Lemma 3.17 Structure of $\mathrm{Res}_\sigma R[G/H,-]_{\mathcal{M}_\mathcal{F}}$**.** [MPN06, Proposition 3.6]
There is an $\mathcal{O}_\mathcal{F}$-module isomorphism:

$$\mathrm{Res}_\sigma R[G/H,-]_{\mathcal{M}_\mathcal{F}} \cong \bigoplus_{L \leq H} R \otimes_{W_H L} R[G/L,-]_{\mathcal{O}_\mathcal{F}}$$

Example 3.18. If $\underline{R}(-)$ is the constant contravariant $\mathcal{O}_\mathcal{F}$-module then, using Lemma 3.17,

$$\begin{aligned}
\mathrm{Ind}_\sigma \underline{R}(G/H) &\cong R[G/H, \sigma(\not-)]_{\mathcal{M}_\mathcal{F}} \otimes_{\mathcal{O}_\mathcal{F}} \underline{R}(\not-) \\
&\cong \bigoplus_{L \leq H} R \otimes_{W_H L} R[G/L, \not-]_{\mathcal{O}_\mathcal{F}} \otimes_{\mathcal{O}_\mathcal{F}} \underline{R}(\not-) \\
&\cong \bigoplus_{L \leq H} R
\end{aligned}$$

Checking the morphisms as well, one sees that

$$\mathrm{Ind}_\sigma \underline{R}(-) \cong B^G(-)$$

Where $B^G(-)$ is the Burnside functor defined at the beginning of the next section.

As well as the properties of induction and restriction inherited from Proposition 1.20, we have the following crucial result.

Proposition 3.19 [MPN06, Theorem 3.8] Although induction with σ is not exact in general, induction with σ takes contravariant resolutions of $\underline{R}(-)$ by projective $\mathcal{O}_\mathcal{F}$-modules to resolutions of $B^G(-)$ by projective $\mathcal{M}_\mathcal{F}$-modules.

3.3 Homology and Co-homology

As in Section 1.4 we have functors $\mathrm{Ext}^*_{\mathcal{M}_\mathcal{F}}$ and $\mathrm{Tor}^{\mathcal{M}_\mathcal{F}}_*$. Furthermore, we define $H^*_{\mathcal{M}_\mathcal{F}}$ and $H^{\mathcal{M}_\mathcal{F}}_*$ for any $\mathcal{M}_\mathcal{F}$-module $A(-)$ as

$$H^*_{\mathcal{M}_\mathcal{F}}(G, A(\not-)) = \mathrm{Ext}^*_{\mathcal{M}_\mathcal{F}}(B^G(\not-), A(\not-))$$

$$H^{\mathcal{M}_\mathcal{F}}_*(G, A(\not-)) = \mathrm{Tor}^{\mathcal{M}_\mathcal{F}}_*(B^G(\not-), A(\not-))$$

Where $B^G(-)$ is the Burnside functor $B^G(-)$ which, by an abuse of notation since G/G is not an object of $\mathcal{M}_\mathcal{F}$, can be defined as

$$B^G(-) = R[-, G/G]_{\mathcal{M}_\mathcal{F}}$$

Upon evaluation at G/K for some finite K,

$$B^G(G/K) = \bigoplus_{\substack{L \leq K \\ \text{Up to } K\text{-conjugacy}}} R_L$$

This is not so dissimilar from the case of the orbit category $\mathcal{O}_\mathcal{F}$ where, using a similar abuse of notation, one could view $\underline{R}(-)$ as $R[-, G/G]_{\mathcal{O}_\mathcal{F}}$. Note that the constant functor $\underline{R}(-)$ used with $\mathcal{O}_{\mathcal{F}}$-modules is not an $\mathcal{M}_{\mathcal{F}}$-module. Specialising the definitions of Section 1.5, G is said to be $\mathcal{M}_{\mathcal{F}}\,\mathrm{FP}_n$ if there is a projective resolution of $B^G(-)$, finitely generated up to degree n, and G has $\mathcal{M}_{\mathcal{F}}\,\mathrm{cd}\,G \leq n$ if there is a length n projective resolution of $B^G(-)$ by $\mathcal{M}_{\mathcal{F}}$-modules.

A corollary of Proposition 3.19 is the following.

Corollary 3.20 [MPN06, Theorem 3.8]

$$H^n_{\mathcal{M}_{\mathcal{F}}}(G, M(\not-)) \cong H^n_{\mathcal{O}_{\mathcal{F}}}(G, \mathrm{Res}_\sigma M(\not-))$$

3.4 Cohomological Dimension

There are no original results in the this Section, but for completeness we provide a brief overview. In [MPN06], it is proven that $\mathrm{vcd}\,G = \mathcal{M}_{\mathcal{F}}\,\mathrm{cd}\,G$ whenever G is virtually torsion free, and in [Deg13b, 6.2.25] it is proved that whenever G has a bound on the orders of its finite subgroups and $\mathcal{F}\text{-}\mathrm{cd}\,G < \infty$ then $\mathcal{F}\text{-}\mathrm{cd}\,G = \mathcal{M}_{\mathcal{F}}\,\mathrm{cd}\,G$.

Question 3.21. Does the condition $\mathcal{M}_{\mathcal{F}}\,\mathrm{cd}\,G < \infty$ imply that $\mathcal{O}_{\mathcal{F}}\,\mathrm{cd}\,G < \infty$, or that $\mathcal{F}\text{-}\mathrm{cd}\,G = \mathcal{M}_{\mathcal{F}}\,\mathrm{cd}\,G$?

3.5 FP_n Conditions

As far as we are aware, there are no results in the literature on the conditions $\mathcal{M}_{\mathcal{F}}\,\mathrm{FP}_n$. We make some small observations about these conditions in this section.

In the lemmas below, $\mathcal{F}/G$ denotes the poset of conjugacy classes of finite subgroups of G, ordered by subconjugation, so $H \preceq K$ if H is subconjugate to K.

Lemma 3.22 G is $\mathcal{M}_{\mathcal{F}}\,\mathrm{FP}_0$ if and only if $\mathcal{F}/G$ is finite.

The proof needs an easy Lemma.

Lemma 3.23 $\mathcal{F}/G$ has is finite if and only if $\mathcal{F}/G$ has a finite cofinal subset.

Proof. One direction is obvious, for the other direction let M be a finite cofinal subset of $\mathcal{F}/G$. Then every element $K \in M$ has finitely many subconjugate subgroups, and since every finite subgroup of G is subconjugate to some $K \in M$ there can only be finitely many finite subgroups up to conjugation. □

Proof of Lemma 3.22. Let f be an $\mathcal{M}_{\mathcal{F}}$-morphism

$$f : R[-, G/K]_{\mathcal{M}_{\mathcal{F}}} \longrightarrow B^G(-) \cong R[-, G/G]_{\mathcal{M}_{\mathcal{F}}}$$

Firstly, we claim that the element m of $R[G/S, G/G]_{\mathcal{M}_{\mathcal{F}}}$ given by

$$m = \big(G/S \xleftarrow{\mathrm{id}} G/S \longrightarrow G/G\big)$$

cannot be in the image of $f(G/S)$ unless S is subconjugate to K. Assume for a contradiction that S is not subconjugate to K and assume m is in the image of

$f(G/S)$. Thus $m = f(G/S)\varphi$ for some $\varphi \in [G/S, G/K]_{\mathcal{M}_\mathcal{F}}$. Thinking of f as a natural transformation gives the commutative diagram below

$$\begin{array}{ccc} R[G/S, G/K]_{\mathcal{M}_\mathcal{F}} & \xrightarrow{f(G/S)} & R[G/S, G/G]_{\mathcal{M}_\mathcal{F}} \\ \uparrow{\scriptstyle \varphi^*} & & \uparrow{\scriptstyle \varphi^*} \\ R[G/K, G/K]_{\mathcal{M}_\mathcal{F}} & \xrightarrow{f(G/K)} & R[G/K, G/G]_{\mathcal{M}_\mathcal{F}} \end{array}$$

Where

$$\begin{aligned} m &= f(G/S)\varphi \\ &= f(G/S) \circ \varphi^* \, \mathrm{id}_{[G/K,G/K]_{\mathcal{M}_\mathcal{F}}} \\ &\cong \big(\varphi^* \circ f(G/K)\big)(\mathrm{id}_{[G/K,G/K]_{\mathcal{M}_\mathcal{F}}}) \end{aligned}$$

Let $f(G/K)(\mathrm{id}_{[G/K,G/K]_{\mathcal{M}_\mathcal{F}}}) = \sum_i r_i x_i$, where $r_i \in R$ and the x_i are basic morphisms in $R[G/K, G/G]_{\mathcal{M}_\mathcal{F}}$. Similarly, let $\varphi = \sum_j s_j y_j$ for $s_j \in R$ and where the y_j are basic morphisms in $R[G/S, G/K]_{\mathcal{M}_\mathcal{F}}$. By assumption we have that

$$\begin{aligned} m &= \varphi^* \sum_i r_i x_i \\ &= \sum_i r_i x_i \circ \sum_j s_j y_j \\ &= \sum_{i,j} (r_i s_j) x_i \circ y_j \end{aligned}$$

There must exist some i and j for which $x_i \circ y_j$ is a morphism which, when written as a sum of basic morphisms, has one component some multiple of m. We calculate $x_i \circ y_j$ for this i and j. Write x_i and y_j in their standard forms as below,

$$\begin{aligned} x_i &= \Big(G/K \longleftarrow G/L_i \longrightarrow G/G\Big) \\ y_j &= \Big(G/S \longleftarrow G/J_j \longrightarrow G/K\Big) \end{aligned}$$

Their composition is

$$x_i \circ y_j = \sum_k \left(\begin{array}{ccccccc} & & & G/X_k & & & \\ & & \swarrow & & \searrow & & \\ & G/J_j & & & & G/L_i & \\ \swarrow & & \searrow & & \swarrow & & \searrow \\ G/S & & & G/K & & & G/G \end{array} \right)$$

Where X_k is some finite subgroup of G which is subconjugate to both L_i and J_j. But such a finite subgroup cannot be conjugate to S, as L_i is subconjugate to K and K is not subconjugate to S by assumption. This contradicts our earlier assertion that $x_i \circ y_j$ when written as a sum of basic morphisms, has one component some multiple of m. Thus, for m to be in the image of $f(G/S)$, S must be subconjugate to K.

Now, if G is $\mathcal{M}_{\mathcal{F}}\mathrm{FP}_0$ then $B^G(-)$ admits an epimorphism from some finitely generated free

$$\bigoplus_{i\in I} R[-, G/K_i]_{\mathcal{M}_{\mathcal{F}}} \twoheadrightarrow B^G(-)$$

As this set I is finite, the argument above implies that all the finite subgroups of G are subconjugate to one of a finite collection of finite subgroups. Thus there is a finite cofinal subset of $\mathcal{F}/G$, and by Lemma 3.23, $\mathcal{F}/G$ is finite.

For the converse assume that $\mathcal{F}/G$ is finite and let $M \subseteq \mathcal{F}/G$ denote a finite cofinal subset of $\mathcal{F}/G$ (we could just take $M = \mathcal{F}/G$), we claim the augmentation map

$$\varepsilon : \bigoplus_{K\in M} R[-, G/K]_{\mathcal{M}_{\mathcal{F}}} \longrightarrow B^G(-)$$

is an epimorphism. Every basic morphism in $B^G(S) \cong R[G/S, G/G]_{\mathcal{M}_{\mathcal{F}}}$ can be written as

$$m = \big(G/S \xleftarrow{\alpha_1} G/L \xrightarrow{\alpha_1} G/G\big)$$

Let $K \in M$ be a finite subgroup with $L \preceq K$, then m is the image of

$$\big(G/S \xleftarrow{\alpha_1} G/L \xrightarrow{\alpha_1} G/K\big) \in R[G/S, G/K]_{\mathcal{M}_{\mathcal{F}}}$$

under the map $R[G/S, G/K]_{\mathcal{M}_{\mathcal{F}}} \longrightarrow B^G(G/S)$. □

Lemma 3.24 If G is $\mathcal{M}_{\mathcal{F}}\,\mathrm{FP}_n$ then G is FP_n.

Proof. Let $P_*(-)$ be a projective resolution of $B^G(-)$ by $\mathcal{M}_{\mathcal{F}}$-modules. Then evaluating at $G/1$ gives a resolution of R by RG modules of type FP_∞ by Lemma 3.12. A dimension shifting argument, as in the last paragraph of Proposition 2.34, completes the proof. □

Lemma 3.25 If G is $\mathcal{O}_{\mathcal{F}}\,\mathrm{FP}_n$ then G is $\mathcal{M}_{\mathcal{F}}\,\mathrm{FP}_n$.

Proof. We use the Bieri-Eckmann criterion (Theorem 1.28). Let $M_\lambda(-)$ be a directed system of $\mathcal{M}_{\mathcal{F}}$-modules such that $\varinjlim M_\lambda(-) = 0$. Then by Corollary 3.20,

$$\varinjlim H^n_{\mathcal{M}_{\mathcal{F}}}(G, M_\lambda(-)) \cong \varinjlim H^n_{\mathcal{O}_{\mathcal{F}}}(G, \mathrm{Res}_\sigma\, M_\lambda(-))$$

Since G is assumed $\mathcal{O}_{\mathcal{F}}\,\mathrm{FP}_n$ the right hand side is 0 by Theorem 1.28, and by another application of the same theorem G is $\mathcal{M}_{\mathcal{F}}\,\mathrm{FP}_n$. □

Recal that G has $\mathcal{O}_{\mathcal{F}}\mathrm{FP}_0$ if and only if G has finitely many conjugacy classes of finite subgroups (Proposition 2.33), thus the conditions $\mathcal{M}_{\mathcal{F}}\mathrm{FP}_0$ and $\mathcal{O}_{\mathcal{F}}\mathrm{FP}_0$ are equivalent. We have the chain of implications

$$\mathcal{O}_{\mathcal{F}}\,\mathrm{FP}_n \Rightarrow \mathcal{M}_{\mathcal{F}}\mathrm{FP}_n \Rightarrow \big(\mathrm{FP}_n + \mathcal{O}_{\mathcal{F}}\mathrm{FP}_0\big)$$

Except for the case $n = 0$ we do not know if the arrows are reversible, although examples in [LN03] show that $\mathrm{FP}_n + \mathcal{O}_{\mathcal{F}}\mathrm{FP}_0 \not\Rightarrow \mathcal{O}_{\mathcal{F}}\,\mathrm{FP}_n$ in general so at least one of the arrows must be irreversible.

Question 3.26. Is there a nice characterisation of the condition $\mathcal{M}_{\mathcal{F}}\,\mathrm{FP}_n$?

3.6 Questions

Collected here are questions related to Mackey functors discussed in this section.

Question 3.21. Does the condition $\mathcal{M}_{\mathcal{F}} \operatorname{cd} G < \infty$ imply that $\mathcal{O}_{\mathcal{F}} \operatorname{cd} G < \infty$, or that $\mathcal{F}\text{-}\operatorname{cd} G = \mathcal{M}_{\mathcal{F}} \operatorname{cd} G$?

Question 3.26. Is there a nice characterisation of the condition $\mathcal{M}_{\mathcal{F}} \operatorname{FP}_n$?

4 Cohomological Mackey Functors

A Mackey functor is called cohomological if, using the language of Remark 3.9, it satifies

$$M(I_K^H) \circ M(R_K^H) = |H : K|$$

for all finite subgroups $K \leq H$ of G. Recall from Remark 3.9 that to describe a Mackey functor $M(-)$ it is sufficient to describe it on objects and on the induction, restriction and conjugation morphisms in $\mathcal{M}_{\mathcal{F}}$ (I_K^H, R_K^H and c_g), we use this in the examples below.

Example 4.1. Group cohomology.

The group cohomology functor is Mackey, more precisely the functor

$$H^n(-, R) : G/H \longmapsto H^n(H, R)$$

Where $H^n(-, R)(c_g)$ is induced by conjugation, $H^n(-, R)(R_K^H)$ is the usual restriction map and $H^n(-, R)(I_K^H)$ is the transfer (see for example [Bro94, §III.9]). That the group cohomology functor satisfies (M) is [Bro94, III.9.5(ii)]. In fact, cohomological Mackey functors get their name from the group cohomology functors.

Example 4.2. Fixed point and fixed quotient functors.

If M is a $\mathbb{Z}G$-module then we write M^- for the fixed point functor

$$M^- : G/H \longmapsto M^H$$

Where $M^H = \mathrm{Hom}_{RH}(R, M)$. For any finite subgroups $K \leq H$ of G, $M^-(R_K^H)$ is the inclusion, $M^-(I_K^H)$ is the trace $m \mapsto \sum_{h \in H/K} hm$, and $M^-(c_g)$ is the map $m \mapsto gm$.

We write M_- for the fixed quotient functor

$$M_- : G/H \longmapsto M_H$$

Where $M_H = R \otimes_{RH} M$. For any finite subgroups $K \leq H$ of G, $M_-(R_K^H)$ is the trace $1 \otimes m \mapsto 1 \otimes \sum_{h \in H/K} hm$, $M_-(I_K^H)$ is the inclusion, and $M_-(c_g)$ is the map $m \mapsto gm$.

Lemma 4.3 [MPN06, Lemma 4.2][TW90, 6.1] There are Mackey functor isomorphisms for any RG-module M,

$$\mathrm{CoInd}_{RG}^{\mathcal{M}_{\mathcal{F}}} M \cong M^-$$

$$\mathrm{Ind}_{RG}^{\mathcal{M}_{\mathcal{F}}} M \cong M_-$$

Where induction and coinduction are with the functor $RG \to \mathcal{M}_{\mathcal{F}}$ given by composition of the usual inclusion functor $RG \to \mathcal{O}_{\mathcal{F}}$ and the functor $\sigma : \mathcal{O}_{\mathcal{F}} \to \mathcal{M}_{\mathcal{F}}$ of Section 3.2. Thus there are also adjoint isomorphisms, for any Mackey functor $N(-)$.

$$\mathrm{Hom}_{RG}(N(G/1), M) \cong \mathrm{Hom}_{\mathcal{M}_{\mathcal{F}}}(N, M^-)$$

$$\mathrm{Hom}_{RG}(M, N(G/1)) \cong \mathrm{Hom}_{\mathcal{M}_{\mathcal{F}}}(M^-, N)$$

As observed by Thévenaz and Webb in [TW95, §16], in [Yos83] Yoshida proves that the category of cohomological Mackey modules is isomorphic to the category of modules over the Hecke category $\mathcal{H}_{\mathcal{F}}$, which we shall describe below. Yoshida concentrates mainly on finite groups but observes in [Yos83, §5, Theorem 4.3′] that this isomorphism will hold for $\mathcal{M}_{\mathcal{F}}$ modules, where $\mathcal{F}$ is any subfamily of the family of finite groups.

The Hecke category $\mathcal{H}_{\mathcal{F}}$, or $\mathcal{H}_{\mathcal{F}}$G if we want to emphasize the group, has for objects the transitive G-sets with finite stabilisers G/H. The morphisms between the objects G/H and G/K are exactly the RG module homomorphisms, $\mathrm{Hom}_{RG}(R[G/H], R[G/K])$. We will use the notation of Section 1, writing $R[G/H, G/K]_{\mathcal{H}_{\mathcal{F}}}$ to denote the morphisms in $\mathcal{H}_{\mathcal{F}}$ between G/H and G/K. Modules over $\mathcal{H}_{\mathcal{F}}$ are also defined as in Section 1, as contravariant functors $\mathcal{H}_{\mathcal{F}} \to$ **R-Mod**. As discussed in Section 1.2 the free $\mathcal{H}_{\mathcal{F}}$-modules are direct sums of modules of the form $R[-, G/H]_{\mathcal{H}_{\mathcal{F}}}$.

Remark 4.4. The usual definition of the Hecke category, for example in [Yos83] and [Tam89], takes the objects of $\mathcal{H}_{\mathcal{F}}$ to be the permutation modules $R[G/H]$ for H finite and the same morphism sets. This is equivalent to our definition above. We choose to take the G-sets G/H as objects so that our notation for modules over $\mathcal{H}_{\mathcal{F}}$ coincides with that for modules over $\mathcal{O}_{\mathcal{F}}$ and $\mathcal{M}_{\mathcal{F}}$.

Remark 4.5. In [Deg13a] and [Deg13b], Degrijse considers categories called $\mathrm{Mack}_{\mathcal{F}}G$ and $\mathrm{coMack}_{\mathcal{F}}G$. In the notation used here $\mathrm{Mack}_{\mathcal{F}}G$ is the category of $\mathcal{M}_{\mathcal{F}}$-modules and $\mathrm{coMack}_{\mathcal{F}}G$ is the subcategory of cohomological Mackey functors, he doesn't study modules over $\mathcal{H}_{\mathcal{F}}$ explicitly.

Thévenaz and Webb also describe a map $\pi : \mathcal{M}_{\mathcal{F}} \to \mathcal{H}_{\mathcal{F}}$ (they call this map α), taking objects G/H in $\mathcal{M}_{\mathcal{F}}$ to their associated permutation modules $R[G/H]$ and morphisms which they describe as follows, for any $K \leq H$,

- $\pi(R^H_K)$ is the natural projection map $R[G/K] \to R[G/H]$.
- $\pi(I^H_K)$ takes $gH \mapsto \sum_{h\in H/K} ghK$.
- $\pi(c_x)$ takes $gH \mapsto gxH^x$.

If $M(-)$ is an $\mathcal{H}_{\mathcal{F}}$ module then it is straightforward to check that $M \circ \pi(-)$ is a $\mathcal{M}_{\mathcal{F}}$-module, see for example [Tam89, p.809] for a proof. Moreover, every cohomological Mackey functor $M(-) : \mathcal{M}_{\mathcal{F}} \to$ **R-Mod** factors through the map π, this is the main result in [Yos83], see also [Web00, §7]. Thus we may pass freely between cohomological Mackey functors and modules over $\mathcal{H}_{\mathcal{F}}$.

Lemma 4.6 [Yos83, Lemma 3.1′] There is an isomorphism, for any finite subgroups H and K of G

$$R[H\backslash G/K] \cong R[G/H, G/K]_{\mathcal{H}_{\mathcal{F}}}$$

Under this identification, morphism composition is given by

$$(HxK) \cdot (KyL) = \sum_{z\in H\backslash G/L} |(HxK \cap zLy^{-1}K)/K| \, (HzL)$$

Remark 4.7. The identification in the lemma above relates to the usual defin-tion of $R[G/H, G/K]_{\mathcal{H}_\mathcal{F}}$ as $\mathrm{Hom}_{RG}(R[G/H], R[G/H])$ with the isomorphism

$$\psi : R[H\backslash G/K] \xrightarrow{\cong} \mathrm{Hom}_{RG}(R[G/H], R[G/K])$$
$$HxK \longmapsto \left(gH \longmapsto \sum_{u \in H/(H \cap xKx^{-1})} guxK \right)$$

Notice that ψ satisfies

$$\psi([HxK] \cdot [KxL]) = \psi([KxL]) \circ \psi([HxK])$$

Remark 4.8. Explicit Description of π. Using the identification of Lemma 4.6, for any $K \leq H$, we can describe π as follows.

- $\pi(R^H_K) = KH$, since according to Lemma 4.6, KH corresponds to the morphism $gK \mapsto gH$, which is exactly Thévenaz and Webb's description of $\pi(R^H_K)$.
- $\pi(I^H_K) = HK$, as according to Lemma 4.6, HK corresponds to the morphism $gH \mapsto \sum_{u \in H/K} uK$, which is Thévenaz and Webb's description of $\pi(I^H_K)$.
- $\pi(c_x) = HxH^x$, similarly to the above because HxH^x corresponds to the morphism $gH \mapsto gxH^x$.

Lemma 4.9 Free and projective $\mathcal{H}_\mathcal{F}$-modules.[TW95, Theorem 16.5(ii)]
The free $\mathcal{H}_\mathcal{F}$-modules are exactly the fixed point functors of permutation modules with finite stabilisers, and the projective $\mathcal{H}_\mathcal{F}$-modules are exactly the fixed point functors of direct summands of permutation modules with finite stabilisers.

4.1 Induction

In this Section we specialise the results of Section 1.3 to the category of cohomological Mackey functors. The main result of this Section will be Proposition 4.14, that we may induce projective resolutions of $\mathcal{O}_\mathcal{F}$-modules to projective resolutions of $\mathcal{H}_\mathcal{F}$-modules.

Let π denote the functor $\pi : \mathcal{M}_\mathcal{F} \to \mathcal{H}_\mathcal{F}$ discussed at the beginning of this Section, and recall from Section 3.2 that $\sigma : \mathcal{O}_\mathcal{F} \to \mathcal{M}_\mathcal{F}$ is the covariant inclusion functor, taking a G-map

$$\alpha_x : G/H \longrightarrow G/K$$
$$H \mapsto xK$$

to the element

$$\sigma\alpha_x = \left(G/H \xleftarrow{\alpha_1} G/H \xrightarrow{\alpha_x} G/K\right)$$
$$= c_x \circ R^{K^{x^{-1}}}_H$$

We need three lemmas leading us to Proposition 4.14.

Lemma 4.10 There is an $\mathcal{O}_\mathcal{F}$-module isomorphism,

$$\operatorname{Res}_{\pi\circ\sigma} R[G/L,-]_{\mathcal{H}_\mathcal{F}} \cong \operatorname{Hom}_{RL}(R, R[G/1,-]_{\mathcal{O}_\mathcal{F}})$$

Proof. Let H be a finite subgroup, evaluating the left hand side at G/H yields $R[G/L,G/H]_{\mathcal{H}_\mathcal{F}}$ while evaluating the right hand side at G/H yields

$$\begin{aligned}\operatorname{Hom}_{RL}(R, R[G/H]) &\cong \operatorname{Hom}_{RG}(RG \otimes_{RL} R, R[G/H]) \\ &\cong \operatorname{Hom}_{RG}(R[G/L], R[G/H]) \\ &\cong R[G/L, G/H]_{\mathcal{H}_\mathcal{F}}\end{aligned}$$

Where the first isomorphism is [Bro94, p.63 (3.3)]. If $\alpha_x : G/H \to G/K$ is the G-map $H \mapsto xK$ then looking at the left hand side

$$\begin{aligned}\operatorname{Res}_{\pi\circ\sigma} R[G/L,-]_{\mathcal{H}_\mathcal{F}}(\alpha_x) &= R[G/H,-]_{\mathcal{H}_\mathcal{F}}(c_x \circ R_H^{K^{x^{-1}}}) \\ &\cong R[G/H,-]_{\mathcal{H}_\mathcal{F}}(c_x) \circ R[G/H,-]_{\mathcal{H}_\mathcal{F}}(R_H^{K^{x^{-1}}})\end{aligned}$$

But $R[G/H,-]_{\mathcal{H}_\mathcal{F}}(R_H^{K^{x^{-1}}})$ is post-composition with the G-map

$$\alpha_1 : G/H \to G/K^{x^{-1}}$$

and $R[G/H,-]_{\mathcal{H}_\mathcal{F}}(c_x)$ is post-composition with the G-map

$$\alpha_x : G/K^{x^{-1}} \to G/K$$

In summary, $\operatorname{Res}_{\pi\circ\sigma} R[G/L,-]_{\mathcal{H}_\mathcal{F}}(\alpha_x)$ is the map

$$\begin{aligned}\operatorname{Hom}_{RG}(R[G/L], R[G/H]) &\longrightarrow \operatorname{Hom}_{RG}(R[G/L], R[G/K]) \\ f &\longmapsto \alpha_x \circ f\end{aligned}$$

Since this is $\operatorname{Hom}_{RL}(R, R[G/1,-]_{\mathcal{O}_\mathcal{F}})(\alpha_x)$ also, the left and right hand sides agree on morphisms. □

Lemma 4.11 Let $N(-)$ be an arbitrary projective contravariant $\mathcal{O}_\mathcal{F}$-module and H a finite subgroup of G. Then there is an isomorphim:

$$N(-) \otimes_{\mathcal{O}_\mathcal{F}} \operatorname{Res}_{\pi\circ\sigma} R[G/H,-]_{\mathcal{H}_\mathcal{F}} \cong \operatorname{Hom}_{RH}(R, N(G/1))$$

Before we prove this we need the following.

Lemma 4.12 For any finite subgroup H of G, the module $\operatorname{Ind}_{\mathbb{Z}G}^{\mathcal{O}_\mathcal{F}\mathrm{G}} \operatorname{Ind}_{\mathbb{Z}H}^{\mathbb{Z}G} R(-)$ is of type $\mathcal{O}_\mathcal{F}\,\mathrm{FP}_\infty$.

Proof. Unfortunately we can't use Proposition 2.34 as G is not assumed to be of type $\mathcal{O}_\mathcal{F}\,\mathrm{FP}_0$. Using Lemma 2.8 and the fact that R is FP_∞ as a $\mathbb{Z}H$ module, $\operatorname{Ind}_{\mathbb{Z}H}^{\mathbb{Z}G} R$ is of type FP_∞ over $\mathbb{Z}G$. Choose a finite type free resolution F_* of $\operatorname{Ind}_{\mathbb{Z}H}^{\mathbb{Z}G} R$ by $\mathbb{Z}G$ modules, then $\operatorname{Ind}_{\mathbb{Z}G}^{\mathcal{O}_\mathcal{F}\mathrm{G}} F_*(-)$ is clearly a complex of finitely generated free $\mathcal{O}_\mathcal{F}$G modules. By Proposition 2.13(ii) $\operatorname{Ind}_{\mathbb{Z}G}^{\mathcal{O}_\mathcal{F}}$ is exact, so $\operatorname{Ind}_{\mathbb{Z}G}^{\mathcal{O}_\mathcal{F}} F_*(-)$ is a resolution of $\operatorname{Ind}_{\mathbb{Z}G}^{\mathcal{O}_\mathcal{F}\mathrm{G}} \operatorname{Ind}_{\mathbb{Z}H}^{\mathbb{Z}G} R(-)$ by finitely generated free $\mathcal{O}_\mathcal{F}$G-modules.

□

Proof of Lemma 4.11. The adjointness of induction and restriction gives an isomorphism of $\mathcal{O}_\mathcal{F}$G-modules, for any $\mathcal{O}_\mathcal{F}$G-module $N(-)$,

$$\begin{aligned}\operatorname{Hom}_{RH}(R, N(G/1)) &\cong \operatorname{Hom}_{RG}(\operatorname{Ind}_{\mathbb{Z}H}^{\mathbb{Z}G} R, N(G/1)) \\ &\cong \operatorname{Mor}_{\mathcal{O}_\mathcal{F}\mathrm{G}}(\operatorname{Ind}_{\mathbb{Z}G}^{\mathcal{O}_\mathcal{F}\mathrm{G}} \operatorname{Ind}_{\mathbb{Z}H}^{\mathbb{Z}G} R(\not{-}), N(\not{-}))\end{aligned}$$

There is a chain of isomorphisms,

$$\begin{aligned}&N(\not{-}) \otimes_{\mathcal{O}_\mathcal{F}\mathrm{G}} \operatorname{Res}_{\pi\circ\sigma} R[G/H, \not{-}]_{\mathcal{H}_\mathcal{F}\mathrm{G}} \\ &\quad\cong N(\not{-}) \otimes_{\mathcal{O}_\mathcal{F}\mathrm{G}} \operatorname{Hom}_{RH}(R, R[G/1, \not{-}]_{\mathcal{O}_\mathcal{F}\mathrm{G}}) \\ &\quad\cong N(\not{-}) \otimes_{\mathcal{O}_\mathcal{F}\mathrm{G}} \operatorname{Mor}_{\mathcal{O}_\mathcal{F}\mathrm{G}}(\operatorname{Ind}_{\mathbb{Z}G}^{\mathcal{O}_\mathcal{F}\mathrm{G}} \operatorname{Ind}_{\mathbb{Z}H}^{\mathbb{Z}G} R(\not{\cdot}), R[\not{\cdot}, \not{-}]_{\mathcal{O}_\mathcal{F}\mathrm{G}}) \\ &\quad\cong \operatorname{Mor}_{\mathcal{O}_\mathcal{F}\mathrm{G}}(\operatorname{Ind}_{\mathbb{Z}G}^{\mathcal{O}_\mathcal{F}\mathrm{G}} \operatorname{Ind}_{\mathbb{Z}H}^{\mathbb{Z}G} R(\not{\cdot}), N(\not{\cdot})) \\ &\quad\cong \operatorname{Hom}_{RH}(R, N(G/1))\end{aligned}$$

Where the first isomorphism is Lemma 4.10 and the second and fourth are the adjoint isomorphism mentioned above. The third isomorphism is from Lemma 2.41 for which we need that $\operatorname{Ind}_{\mathbb{Z}G}^{\mathcal{O}_\mathcal{F}} \operatorname{Ind}_{\mathbb{Z}H}^{\mathbb{Z}G} R(-)$ is finitely generated, but this is implied by Lemma 4.12. □

Recall the fixed point functors defined in Example 4.2. The fixed point functor R^- can be described explicitly as $R^H = R$ for all finite H, and on morphisms,

$$\begin{aligned}R^-(R_K^H) &= \mathrm{id}_R \\ R^-(I_K^H) &= (r \mapsto |H : K|r) \\ R^-(c_g) &= \mathrm{id}_R\end{aligned}$$

Lemma 4.13 $\operatorname{Ind}_{\pi\circ\sigma} \underline{R}_{\mathcal{O}_\mathcal{F}}(-) \cong R^-$

Proof. The proof is split into two parts, first we check that the two functors agree on objects, then we check they agree on morphisms. Throughout the proof H, K and L will be finite subgroups of G. If $\alpha : G/L \to G/K$ is a G-map then we will write α_* for the induced map

$$\alpha_* : \operatorname{Hom}_{RG}(R[G/H], R[G/L]) \longrightarrow \operatorname{Hom}_{RG}(R[G/H], R[G/K])$$

and also for the induced map

$$\alpha_* : R[H\backslash G/L]) \longrightarrow R[H\backslash G/K]$$

where $R[H\backslash G/L])$ is identified with $\operatorname{Hom}_{RG}(R[G/H], R[G/L])$ using the isomorphism ψ of Remark 4.7. Note that with this notation, $\alpha_* \circ \psi = \psi \circ \alpha_*$.
The functors $\operatorname{Ind}_{\pi\circ\sigma} \underline{R}_{\mathcal{O}_\mathcal{F}}(-)$ *and* R^- *agree on objects:*

For any finite subgroup H of G,

$$\begin{aligned}
\mathrm{Ind}_{\pi\circ\sigma}\underline{R}_{\mathcal{O}_\mathcal{F}}(G/H) &= \underline{R}(?) \otimes_{\mathcal{O}_\mathcal{F}} R[G/H, \pi\circ\sigma(?)]_{\mathcal{H}_\mathcal{F}} \\
&\cong \underline{R}_{\mathcal{O}_\mathcal{F}}(?) \otimes_{\mathcal{O}_\mathcal{F}} \mathrm{Hom}_{RG}(R[G/H], R[G/1, ?]_{\mathcal{O}_\mathcal{F}}) \\
&\cong \bigoplus_{K\in\mathcal{F}} R\otimes_R \mathrm{Hom}_{RG}(R[G/H], R[G/K]) \Big/ \substack{\alpha^*1\otimes x_K \sim 1\otimes \alpha_* x_L \text{ for} \\ \alpha : G/L\to G/K \text{ any } G \text{ map}, \\ x_K\in\mathrm{Hom}_{RG}(R[G/H],R[G/K]) \\ x_L\in\mathrm{Hom}_{RG}(R[G/H],R[G/L])} \\
&\cong \bigoplus_{K\in\mathcal{F}} \mathrm{Hom}_{RG}(R[G/H], R[G/K]) \Big/ {x_K\sim\alpha_* x_L} \\
&\cong \bigoplus_{K\in\mathcal{F}} R[H\backslash G/K] \Big/ \substack{[HxL]\sim\alpha_*[HxL] \\ \alpha : G/L\to G/K \text{ any } G \text{ map.}}
\end{aligned}$$

Where the first isomorphism is Lemma 4.10 and the last is Lemma 4.6. Let $HxK \in R[H\backslash G/K]$ be an arbitrary element, and consider the G-map

$$\begin{aligned}
\alpha_x : G/(H\cap K^{x^{-1}}) &\longrightarrow G/K \\
(H\cap K^{x^{-1}}) &\longmapsto xK
\end{aligned}$$

Then

$$\begin{aligned}
\psi\Big((\alpha_x)_*\big(H1(H\cap K^{x^{-1}})\big)\Big) &= (\alpha_x)_*\left(H\longmapsto \sum_{h\in H/(H\cap K^{x^{-1}})} hK^{x^{-1}}\right) \\
&= \left(H\longmapsto \sum_{h\in H/(H\cap K^{x^{-1}})} hxK\right) \\
&= \psi\,(HxK)
\end{aligned}$$

Thus, in $\mathrm{Ind}_{\pi\circ\sigma}\,\underline{R}_{\mathcal{O}_\mathcal{F}}(G/H)$, the elements $[HxK]$ and $[H1(H\cap K^{x^{-1}})]$ are equal, where $[-]$ denotes an equivalence class of elements under the relation $\sim$. So we can write

$$\mathrm{Ind}_{\pi\circ\sigma}\,\underline{R}_{\mathcal{O}_\mathcal{F}}(G/H) \cong \bigoplus_{\substack{K\in\mathcal{F}\\K\le H}} R[H\backslash G/K] \Big/ \substack{[HxL]\sim\alpha_*[HxL] \\ \alpha : G/L\to G/K \text{ any } G \text{ map.} \\ L\in\mathcal{F},\ L\le H}$$

Next, we show that if $K\le H$ then $[H1K] = [|H:K|H1H]$. Let $\alpha_1 : G/K \to G/H$ be the projection. Then

$$\begin{aligned}
\psi\left((\alpha_1)_*(H1K)\right) &= (\alpha_1)_*\left(H\longmapsto \sum_{h\in H/K} hK\right) \\
&= \big(H\longmapsto |H:K|H\big) \\
&= \psi\left(|H:K|(H1H)\right)
\end{aligned}$$

Combining the two facts proved above,

$$[HxK] = |H : H\cap K^{x^{-1}}|\,[H1H] \qquad (\star)$$

In particular, any element $[HxK]$ is equal to some multiple of $[H1H]$, so

$$\mathrm{Ind}_{\pi\circ\sigma}\,\underline{R}_{\mathcal{O}_\mathcal{F}}(G/H) \cong R$$

Showing the two functors $\mathrm{Ind}_{\pi\circ\sigma}\,\underline{R}_{\mathcal{O}_\mathcal{F}}(-)$ and R^- agree on objects.
The functors $\mathrm{Ind}_{\pi\circ\sigma}\,\underline{R}_{\mathcal{O}_\mathcal{F}}(-)$ *and* R^- *agree on morphisms:*

Following the generator $[H1H]$ up the chain of isomorphisms at the beginning of the proof shows the element

$$1 \otimes \mathrm{id}_{R[G/H]} \in R(?) \otimes_{\mathcal{O}_\mathcal{F}} R[G/H, \pi\circ\sigma(?)]_{\mathcal{H}_\mathcal{F}}$$

generates $\mathrm{Ind}_{\pi\circ\sigma}\,\underline{R}_{\mathcal{O}_\mathcal{F}}(G/H) \cong R$, where

$$\mathrm{id}_{R[G/H]} \in \mathrm{Hom}_{RG}(R[G/H], R[G/H]) \cong R[G/H, G/H]_{\mathcal{H}_\mathcal{F}}$$

Now, for some finite subgroup K with $K \le H$,

$$\mathrm{Ind}_{\pi\circ\sigma}\,\underline{R}_{\mathcal{O}_\mathcal{F}}(R^H_K) : 1 \otimes \mathrm{id}_{R[G/H]} \mapsto 1 \otimes \pi$$

Where $\pi : R[G/K] \mapsto R[G/H]$ is the projection map. Following this back down the chain of isomorphisms at the beginning of the proof, gives the element $[K1H]$. Using $(\star)$, $[K1H] = [K1K]$, so $\mathrm{Ind}_{\pi\circ\sigma}\,\underline{R}_{\mathcal{O}_\mathcal{F}}(R^H_K)$ is the identity on R, as required.

Similarly, for some finite subgroup K with $H \le L$, we calculate

$$\mathrm{Ind}_{\pi\circ\sigma}\,\underline{R}_{\mathcal{O}_\mathcal{F}}(I^L_H) : 1 \otimes \mathrm{id}_{R[G/H]} \longmapsto 1 \otimes t_{L/H}$$

Where $t_{L/H} \in \mathrm{Hom}_{RG}(R[G/L], R[G/H])$ denotes the map $L \mapsto \sum_{l\in L/H} lH$. Following this element back down the chain of isomorphisms we get the element $[L1H]$, which by $(\star)$ is equal to $|L : H|[H1H]$. Thus $\mathrm{Ind}_{\pi\circ\sigma}\,\underline{R}_{\mathcal{O}_\mathcal{F}}(I^L_H)$ acts as multiplication by $|L : H|$ on R, as required.

For any element $x \in G$, we calculate

$$\mathrm{Ind}_{\pi\circ\sigma}\,\underline{R}_{\mathcal{O}_\mathcal{F}}(c_x) : 1 \otimes \mathrm{id}_{R[G/H]} \longmapsto 1 \otimes \gamma_x$$

Where $\gamma_x \in \mathrm{Hom}_{RG}(R[G/H^{x^{-1}}], R[G/H])$ is the map $H^{x^{-1}} \mapsto xH$. Following this down the chain of isomorphisms we get the element $[H^{x^{-1}}xH]$, which by $(\star)$ is equal to $[H^{x^{-1}}1H^{x^{-1}}]$. Thus $\mathrm{Ind}_{\pi\circ\sigma}\,\underline{R}_{\mathcal{O}_\mathcal{F}}(c_x)$ acts as the identity on R, as required. □

The following proposition should be compared with Proposition 3.19.

Proposition 4.14 Induction with $\pi\circ\sigma$ takes projective resolutions of $\underline{R}_{\mathcal{O}_\mathcal{F}}(-)$ by $\mathcal{O}_\mathcal{F}$ modules to projective resolutions of R^- by $\mathcal{H}_\mathcal{F}$ modules.

Proof. Let $P_*(-)$ be a projective resolution of $\underline{R}_{\mathcal{O}_\mathcal{F}}(-)$ by $\mathcal{O}_\mathcal{F}$-modules, then by Lemma 4.11,

$$\begin{aligned}\mathrm{Ind}_{\pi\circ\sigma} P_*(G/H) &= P_*(?) \otimes_{\mathcal{O}_\mathcal{F}} \mathrm{Res}_{\pi\circ\sigma} R[G/H, ?]_{\mathcal{H}_\mathcal{F}} \\ &\cong \mathrm{Hom}_{RH}(R, P_*(G/1))\end{aligned}$$

Inducing $P_*(-) \twoheadrightarrow \underline{R}_{\mathcal{O}_\mathcal{F}}(-)$ with $\pi \circ \sigma$ and using Lemma 4.13 gives the chain complex

$$\operatorname{Ind}_{\pi\circ\sigma} P_*(-) \twoheadrightarrow R^-$$

Induction preserves projectives (Proposition 1.20), so we must show only that the above is exact. Since induction is right exact, it is necessarily exact at position -1 and position 0. Evaluating at G/H gives the resolution

$$\operatorname{Hom}_{RH}(R, P_*(G/1)) \longrightarrow R \qquad (\star)$$

By [Nuc00, Theorem 3.2], the resolution $P_*(G/1)$ splits when restricted to a complex of RH-modules for any finite subgroup H of G. Since $\operatorname{Hom}_{RH}(R,-)$ preserves the exactness of RH-split complexes, $\operatorname{Hom}_{RH}(R, P_*(G/1))$ is exact at position i for all $i \geq 1$, completing the proof. □

Remark 4.15. The Proposition above doesn't hold with $\underline{R}_{\mathcal{O}_\mathcal{F}}(-)$ replaced by an arbitrary $\mathcal{O}_\mathcal{F}$-module $M(-)$, as any resolution of $M(-)$ by projective $\mathcal{O}_\mathcal{F}$-modules will not in general split when evaluated at $G/1$.

4.2 Homology and Co-homology

$\operatorname{Ext}^*_{\mathcal{H}_\mathcal{F}}$ and $\operatorname{Tor}_*^{\mathcal{H}_\mathcal{F}}$ are defined as in Section 1.4, and the homology and cohomology functors are defined as follows, for any $\mathcal{H}_\mathcal{F}$ module $M(-)$,

$$H^*_{\mathcal{H}_\mathcal{F}}(G, M(\not-)) \cong \operatorname{Ext}^*_{\mathcal{H}_\mathcal{F}\mathrm{G}}(R^{\not-}, M(\not-))$$

$$H_*^{\mathcal{H}_\mathcal{F}}(G, M(\not-)) \cong \operatorname{Tor}_*^{\mathcal{H}_\mathcal{F}\mathrm{G}}(M(\not-), R^{\not-})$$

There is the following analog of Corollary 3.20:

Proposition 4.16 For any cohomological Mackey functor $M(-)$,

$$H^n_{\mathcal{H}_\mathcal{F}}(G, M(\not-)) = H^n_{\mathcal{O}_\mathcal{F}}(G, \operatorname{Res}_{\pi\circ\sigma} M(\not-))$$

Proof. Let $P_*(-)$ be a projective $\mathcal{O}_\mathcal{F}$-resolution of $\underline{R}(-)$, then

$$\begin{aligned} H^n_{\mathcal{O}_\mathcal{F}}(G, \operatorname{Res}_{\pi\circ\sigma} M(\not-)) &= H^n \operatorname{Mor}_{\mathcal{O}_\mathcal{F}}(P_*(\not-), \operatorname{Res}_{\pi\circ\sigma} M(\not-)) \\ &\cong H^n \operatorname{Mor}_{\mathcal{H}_\mathcal{F}}(\operatorname{Ind}_{\pi\circ\sigma} P_*(\not-), M(\not-)) \\ &= H^n_{\mathcal{H}_\mathcal{F}}(G, M(\not-)) \end{aligned}$$

Where the isomorphism is adjoint isomorphism between induction and restriction and $\operatorname{Ind}_{\pi\circ\sigma} P_*(\not-)$ is a projective $\mathcal{H}_\mathcal{F}$ resolution of $R^{\not-}$ by Proposition 4.14. □

4.3 FP_n Conditions

The main result of this section is Theorem 4.28 - if G is $\mathcal{H}_\mathcal{F}\,\mathrm{FP}_n$ then G is $\mathcal{F}$-FP_n. For an explanation of relative $\mathcal{F}$-cohomology and the condition $\mathcal{F}$-FP_n see [Nuc99].

Recall from Section 1.5 that an $\mathcal{H}_\mathcal{F}$-module $M(-)$ is finitely generated if and only if there exists a finitely generated free $\mathcal{H}_\mathcal{F}$-module $F(-)$ and an epimorphism $F(-) \twoheadrightarrow M(-)$. An $\mathcal{H}_\mathcal{F}$-module $M(-)$ is said to be $\mathcal{H}_\mathcal{F}\,\mathrm{FP}_n$, for

$n \in \mathbb{N} \cup \{\infty\}$, if there exists a resolution of $M(-)$ by projective modules which is finitely generated in all degrees $\leq n$, and a group G is $\mathcal{H}_{\mathcal{F}}\,\mathrm{FP}_n$ if the $\mathcal{H}_{\mathcal{F}}$-module R^- is $\mathcal{H}_{\mathcal{F}}\,\mathrm{FP}_n$.

Using the same argument as in the proof of Lemma 3.25 gives:

Lemma 4.17 If G is $\mathcal{M}_{\mathcal{F}}\mathrm{FP}_n$ then G is $\mathcal{H}_{\mathcal{F}}\,\mathrm{FP}_n$.

So there is a chain of implications:

$$\mathcal{O}_{\mathcal{F}}\mathrm{FP}_n \Rightarrow \mathcal{M}_{\mathcal{F}}\mathrm{FP}_n \Rightarrow \mathcal{H}_{\mathcal{F}}\,\mathrm{FP}_n \Rightarrow \mathcal{F}\text{-}\mathrm{FP}_n \Rightarrow FP_n + \left\{ \begin{array}{c} G \text{ has finitely many} \\ \text{conjugacy classes} \\ \text{of finite } p\text{-subgroups} \end{array} \right\}$$

Where the final implication is [LN10, Proposition 4.2], where it is proved that G is $\mathcal{F}$-FP_0 if and only if G has finitely many conjugacy classes of finite p-subgroups, for all primes p. It is conjectured in the same paper that a group G of type FP_∞ with finitely many conjugacy classes of finite p subgroups is $\mathcal{F}$-FP_∞ [LN10, Conjecture 4.3].

The implication $\mathcal{H}_{\mathcal{F}}\,\mathrm{FP}_n \Rightarrow \mathcal{F}\text{-}\mathrm{FP}_n$ is not known to be reversible except in the case $n = 0$, which is Proposition 4.29.

Since G is $\mathcal{M}_{\mathcal{F}}\mathrm{FP}_0$ if and only if G has finitely many conjugacy classes of finite subgroups (Lemma 3.22), the implication $\mathcal{M}_{\mathcal{F}}\mathrm{FP}_n \Rightarrow \mathcal{H}_{\mathcal{F}}\,\mathrm{FP}_n$ is not reversible although we don't know if, for example, a group G of type $\mathcal{H}_{\mathcal{F}}\,\mathrm{FP}_n$ and $\mathcal{M}_{\mathcal{F}}\,\mathrm{FP}_0$ is $\mathcal{M}_{\mathcal{F}}\,\mathrm{FP}_n$.

As discussed at the end of Section 3.5, the implication $\mathcal{O}_{\mathcal{F}}\,\mathrm{FP}_n \Rightarrow \mathcal{M}_{\mathcal{F}}\,\mathrm{FP}_n$ is not known to be reversible. There are examples due to Leary and Nucinkis of groups which act properly and cocompactly on contractible G-CW-complexes but which are not of type $\mathcal{O}_{\mathcal{F}}\,\mathrm{FP}_0$ [LN03, Example 3, p.149]. By Remark 4.36, these groups are of type $\mathcal{H}_{\mathcal{F}}\,\mathrm{FP}_\infty$ showing that $\mathcal{H}_{\mathcal{F}}\,\mathrm{FP}_\infty \not\Rightarrow \mathcal{O}_{\mathcal{F}}\,\mathrm{FP}_0$. Leary and Nucinkis also give examples of groups which act properly and cocompactly on contractible G-CW-complexes, are of type $\mathcal{O}_{\mathcal{F}}\,\mathrm{FP}_0$ but which are not $\mathcal{O}_{\mathcal{F}}\,\mathrm{FP}_\infty$ [LN03, Example 4, p.150]. Hence there can be no implication $\mathcal{H}_{\mathcal{F}}\,\mathrm{FP}_n + \mathcal{O}_{\mathcal{F}}\,\mathrm{FP}_0 \not\Rightarrow \mathcal{O}_{\mathcal{F}}\,\mathrm{FP}_n$.

Question 4.18. Are the conditions $\mathcal{H}_{\mathcal{F}}\,\mathrm{FP}_n$ and $\mathcal{F}$-FP_n equivalent?

4.3.1 $\mathcal{H}_{\mathcal{F}}\,\mathrm{FP}_n$ implies $\mathcal{F}$-FP_n

This section comprises a series of lemmas, building to the proof of Theorem 4.28, that the condition $\mathcal{H}_{\mathcal{F}}\,\mathrm{FP}_n$ implies the condition $\mathcal{F}$-FP_n. Throughout, G is a group, and H and K are arbitrary finite subgroups of G.

Lemma 4.19 If

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is a $\mathcal{F}$-split short exact sequence of RG-modules then

$$0 \longrightarrow A^- \longrightarrow B^- \longrightarrow C^- \longrightarrow 0$$

is exact.

Proof. Evaluating the fixed point functor M^- at the finite subgroup H is equivalent to applying the functor $\mathrm{Hom}_{RH}(R, -)$ to M, but since the short exact sequence is split as a sequence of RH-modules this functor is exact. □

We say that a short exact sequence of RG-modules

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \qquad (\star)$$

is *H-good* if

$$0 \longrightarrow A^H \longrightarrow B^H \longrightarrow C^H \longrightarrow 0$$

is exact. Say it is $\mathcal{F}$-good if it is H-good for all finite subgroups H of G.

Remark 4.20. If

$$0 \longrightarrow A^- \longrightarrow B^- \longrightarrow C^- \longrightarrow 0$$

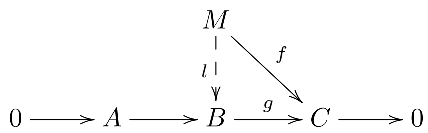
is a short exact sequence of fixed point functors then

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is $\mathcal{F}$-good.

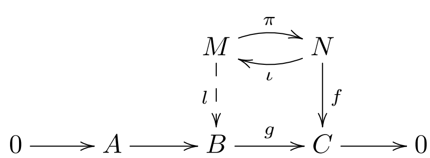
Remark 4.21. By Lemma 4.19, if $(\star)$ is RH-split then it is H-good, however in general being H-good is a weaker property.

Additionally, we say that an RH module M has property (P_H) if for any $\mathcal{F}$-good short exact sequence $(\star)$, $\mathrm{Hom}_{RH}(M, -)$ preserves the exactness of $(\star)$. Since $\mathrm{Hom}_{RH}(M, -)$ is always left exact, having (P_H) is equivalent to asking that for any $\mathcal{F}$-good short exact sequence $(\star)$ and any RH-module homomorphism $f : M \to C$, there is a RH-module homomorphism $l : M \to B$ such that the diagram below commutes.



Lemma 4.22 If M has (P_H) then any direct summand of M, as RH-modules, has (P_H).

Proof. This is, with a minor alteration, the proof of [Rot09, Theorem 3.5(ii)]. Let N be a direct summand of M and consider the diagram with exact bottom row, and assume the bottom row is $\mathcal{F}$-good.



Where f is some arbitrary homomorphism, and π and ι are the projection and inclusion maps respectively. Since M has P_H, there is a map $l : M \to B$ such that $g \circ l = f \circ \pi$, the composition $l \circ \iota$ is the required map. $\square$

Lemma 4.23 As RH-modules,

$$R[G/K] \cong \bigoplus_{g \in H\backslash G/K} R[H/K_g]$$

Where $K_g = \{h \in H \;:\; g^{-1}hg \leq K\}$.

Proof. Consider the action of H on G/K, two elements g_1K and g_2K are in the same H orbit if and only if $Hg_1K = Hg_2K$, and the H-stabiliser of an element gK is the set of $h \in H$ with $hgK = gK$, equivalently $g^{-1}hg \leq K$. □

Lemma 4.24
1. R has (P_H).
2. $R[H/L]$ has (P_H), for L any subgroup of H.
3. $R[G/K]$ has (P_H), for K any subgroup of G.

Proof.
1. The condition that $\mathrm{Hom}_{RH}(R, -)$ preserves an $\mathcal{F}$-good short exact sequence $(\star)$ is exactly the condition that $(\star)$ is H-good, and $\mathcal{F}$-good implies H-good.
2. There are natural isomorphims,

$$\begin{aligned}\mathrm{Hom}_{RH}(R[H/L], -) &\cong \mathrm{Hom}_{RH}(RH \otimes_{RL} R, -)\\ &\cong \mathrm{Hom}_{RL}(R, \mathrm{Hom}_{RH}(RH, -))\\ &\cong \mathrm{Hom}_{RL}(R, -)\end{aligned}$$

   Where the second isomorphism is [Bro94, p.63, (3.3)], now use part (1).
3. Use Lemma 4.23 to rewrite $R[G/K]$ (as an RH-module), as

$$R[G/K] \cong \bigoplus_{g \in H\backslash G/K} R[H/K_g]$$

   Thus

$$\mathrm{Hom}_{RH}(R[G/K], -) \cong \prod_{g \in H\backslash G/K} \mathrm{Hom}_{RH}(R[H/K_g], -)$$

   Now use part (2) and that direct products of exact sequences are exact.

□

Lemma 4.25 If C has (P_H) then $(\star)$ splits as a sequence of RH-modules.

Proof. Apply $\mathrm{Hom}_{RH}(C, -)$ to $(\star)$. □

Lemma 4.26 If P_* is an $\mathcal{F}$-good projective resolution of R, then P_* is $\mathcal{F}$-split.

Proof. Fix a finite subgroup H and let $\partial_i : P_i \to P_{i-1}$ denote the usual boundary map of the chain complex and $\partial_0 : P_0 \to R$ the augmentation map. Consider the short exact sequence

$$0 \longrightarrow \mathrm{Ker}\,\partial_0 \longrightarrow P_0 \longrightarrow R \longrightarrow 0$$

This splits as a sequence of RH modules by Lemmas 4.23(1) and 4.25, and by Lemma 4.22 $\operatorname{Ker}\partial_0$ has (P_H).

This is the base case of an induction which continues as follows: Assume that P_* is shown to split up to degree $i-1$ and $\operatorname{Ker}\partial_{i-1}$ has (P_H), we show it splits in degree $i+1$ also and $\operatorname{Ker}\partial_i$ has (P_H). Consider the short exact sequence

$$\operatorname{Ker}\partial_i \longrightarrow P_i \longrightarrow \operatorname{Ker}\partial_{i-1}$$

Since $\operatorname{Ker}\partial_{i-1}$ has (P_H), Lemma 4.25 shows the short exact sequence splits, and Lemmas 4.24 and 4.22 show that $\operatorname{Ker}\partial_i$ has (P_H). □

Remark 4.27. Similarly to Proposition 4.14, the above Lemma may fail for $\mathcal{F}$-good resolutions of arbitrary modules.

Theorem 4.28 If G is $\mathcal{H}_\mathcal{F}\,\mathrm{FP}_n$ then G is $\mathcal{F}$-FP_n.

Proof. Find a resolution $P_*(-)$ of R^- by finitely generated free $\mathcal{H}_\mathcal{F}$ modules up to dimension n. By Remark 4.20, P_* is an $\mathcal{F}$-good resolution of R by permutation RG modules with finite stabilisers. By Lemma 4.26 P_* is $\mathcal{F}$-split, and by [Nuc99, Definition 2.2] permutation RG modules with finite stabilisers are $\mathcal{F}$-projective. □

We only know of a converse for this theorem in the case $n = 0$:

Proposition 4.29 For any group G, the following conditions are equivalent:

1. G is $\mathcal{H}_\mathcal{F}\,\mathrm{FP}_0$.
2. G is $\mathcal{F}$-FP_0.
3. G has finitely many conjugacy classes of finite p-subgroups, for all primes p.

Proof. $1 \Rightarrow 2$ is given by Theorem 4.28 and $2 \Leftrightarrow 3$ is [LN10, Proposition 4.2]. As part of the proof of [LN10, Proposition 4.2], it is shown that if G has finitely many conjugacy classes of finite p-subgroups then there is an $\mathcal{F}$-split surjection

$$\bigoplus_{\substack{P \leq G \\ P \text{ of prime power order}}} \mathbb{Z}G/P \longrightarrow\!\!\!\!\!\rightarrow \mathbb{Z}$$

Since every $\mathcal{F}$-split surjection is $\mathcal{F}$-good, taking fixed points gives a epimorphism from a finitely generated free $\mathcal{H}_\mathcal{F}$-module onto R^-. □

4.4 Cohomological dimension

The $\mathcal{H}_\mathcal{F}$ cohomological dimension of a group G, denoted $\mathcal{H}_\mathcal{F}\operatorname{cd} G$ is defined to length of the shortest projective resolution of R^- by $\mathcal{H}_\mathcal{F}$-modules, or equivalently

$$\mathcal{H}_\mathcal{F}\operatorname{cd} G = \inf\{n \,:\, H^n_{\mathcal{H}_\mathcal{F}}(G, M(\not -) \neq 0), \text{ where } M(-) \text{ is some } \mathcal{H}_\mathcal{F}\text{-module}\}$$

Remark 4.30. In [Deg13b] the $\mathcal{H}_\mathcal{F}$ cohomological dimension is defined as

$$\mathcal{H}_\mathcal{F}\,\mathrm{cd}\,G = \inf\{n \,:\, H^n_{\mathcal{O}_\mathcal{F}}(G, \mathrm{Res}^{\mathcal{O}_\mathcal{F}}_{\mathcal{H}_\mathcal{F}} M(-) \neq 0), \text{ for some } \mathcal{H}_\mathcal{F}\text{-module } M(-)\}$$

The two definitions are equivalent by Proposition 4.16.

In [Deg13b, 6.2.16], Degrijse shows that for all groups G with $\mathcal{H}_\mathcal{F}\,\mathrm{cd}\,G < \infty$,

$$\mathcal{F}\text{-}\mathrm{cd}\,G = \mathcal{H}_\mathcal{F}\,\mathrm{cd}\,G$$

We can improve this to:

Theorem 4.31 For all groups G,

$$\mathcal{F}\text{-}\mathrm{cd}\,G = \mathcal{H}_\mathcal{F}\,\mathrm{cd}\,G$$

Proof. That $\mathcal{F}\text{-}\mathrm{cd}\,G \leq \mathcal{H}_\mathcal{F}\,\mathrm{cd}\,G$ follows immediately from Remark 4.20 and Lemma 4.26.

For the opposite inequality, we first use [Gan12, Lemma 3.4] which states that for a group G with $\mathcal{F}\text{-}\mathrm{cd}\,G \leq n$ there is an $\mathcal{F}$-projective resolution P_* of $\mathbb{Z}$ of length n, where each P_i is a permutation module with finite stabilisers. Given such a P_*, we take fixed points of P_* to get the $\mathcal{H}_\mathcal{F}$ resolution P_*^-. Since P_* is $\mathcal{F}$-split, P_*^- is exact by Lemma 4.19. □

Proposition 4.32 If G acts properly on a contractible G-CW complex of dimension n then $\mathcal{H}_\mathcal{F}\,\mathrm{cd}\,G \leq n$.

This fact is well known for $\mathcal{F}$-cd instead of $\mathcal{H}_\mathcal{F}\,\mathrm{cd}$, but since a direct proof for $\mathcal{H}_\mathcal{F}\,\mathrm{cd}$ is both interesting and short we provide one.

Proof. Let P_* denote the cellular chain complex for the contractible G-CW-complex X of dimension n and take fixed points to get the complex $P_*^- \longrightarrow R^-$ of $\mathcal{H}_\mathcal{F}$-modules. Since the action of G on X is proper the modules comprising P_* are permutation modules with finite stabilisers and so P_*^- is a chain complex of free $\mathcal{H}_\mathcal{F}$-modules. By a result of Bouc [Bou99] and Kropholler-Wall [KW11] this chain complex splits when restricted to a complex of RH-modules for any finite subgroup H of G. Thus $P_*^H \longrightarrow R$ is exact for any finite subgroup H. □

This leads naturally to the question:

Question 4.33. If $\mathcal{H}_\mathcal{F}\,\mathrm{cd}\,G = n$, does there exist a contractible proper G-CW complex of dimension n?

Brown has conjectured the following:

Question 4.34. [Bro94, VIII.11 p.226] If G is virtually torsion-free with finite virtual cohomological dimension, does there exist a contractible proper G-CW complex of dimension $\mathrm{vcd}\,G$?

If G is virtually torsion free then $\mathrm{vcd}\,G = \mathcal{H}_\mathcal{F}\,\mathrm{cd}\,G$ [MPN06], so a positive answer to Question 4.33 would give a positive answer to Question 4.34 as well.

Related to this is the following question, posed using $\mathcal{F}$-cd instead of $\mathcal{H}_\mathcal{F}\,\mathrm{cd}$ by Nucinkis.

Question 4.35. [Nuc00, p.337] Does $\mathcal{H}_\mathcal{F} \operatorname{cd} G < \infty$ imply that $\mathcal{O}_\mathcal{F} \operatorname{cd} G < \infty$?

Remark 4.36. If G acts properly and cocompactly on a contractible G-CW-complex then, by a modification of the argument of the proof of Lemma 4.32, G is $\mathcal{H}_\mathcal{F} \operatorname{FP}_\infty$ also. However, if G acts properly on a finite type but infinite dimensional complex X, then the Theorem of Bouc and Kropholler-Wall doesn't apply and we do not know if the cellular chain complex of X splits when restricted to a finite subgroup.

Question 4.37. If G acts properly on a contractible G-CW-complex of finite type, but not necessarily finite dimension, then is G of type $\mathcal{H}_\mathcal{F} \operatorname{FP}_\infty$?

4.5 Questions

Collected here are some of the questions related to cohomological Mackey functors from this section.

Question 4.18. Are the conditions $\mathcal{H}_\mathcal{F} \operatorname{FP}_n$ and $\mathcal{F}$-FP_n equivalent?

Question 4.33. If $\mathcal{H}_\mathcal{F} \operatorname{cd} G = n$, does G act properly on a contractible G-CW complex of dimension n?

Question 4.37. If G acts properly on a contractible G-CW-complex of finite type, but not necessarily finite dimension, then is G of type $\mathcal{H}_\mathcal{F} \operatorname{FP}_\infty$?

5 Duality Groups

In [DL03] the notion of Bredon-Poincaré duality groups is first defined and in [MP13, Definition 5.1] this is extended to Bredon-duality groups over arbitrary rings. See [Bie81, Chapter 9] and [Dav00] for the classical case.

Definition 5.1 [MP13, Definition 5.1] A group G is Bredon-duality of dimension n over R if

1. $\mathcal{O}_{\mathcal{F}} \operatorname{cd}_R G = n$.
2. G is $\mathcal{O}_{\mathcal{F}} \operatorname{FP}_\infty$ over R.
3. For all finite subgroups H of G there is an integer n_H such that

$$H^i(WH, R[WH]) = \begin{cases} R\text{-flat} & \text{if } i = n_H \\ 0 & \text{else.} \end{cases}$$

Furthermore, G is Bredon-Poincaré-duality over R if for all finite H,

$$H^{n_H}(WH, R[WH]) = R$$

For torsion-free groups this reduces to the usual definition of duality and Poincaré-duality groups.

We will write $\mathcal{V}$ for the set

$$\mathcal{V} = \{n_F \,:\, F \text{ a non-trivial finite subgroup of } G\} \subseteq \{0, \ldots, n\}$$

In Example 5.54 we will build Bredon duality groups with arbitrary $\mathcal{V}(G)$.

Question 5.2. Is it possible to construct Bredon Poincaré duality groups with prescribed $\mathcal{V}(G)$?

Lemma 5.3
1. If G is Bredon duality of dimension n over $\mathbb{Z}$ then $n_H = \operatorname{cd}_{\mathbb{Q}} WH$ for all finite H, and $n_{\mathrm{id}} \leq n$.
2. If G is R-torsion-free and Bredon duality of dimension n over R then $n_H = \operatorname{cd}_R WH$ and $n_{\mathrm{id}} \leq n$.

To prove the Lemma we need the following proposition, an analog of [Bro94, VIII.6.7] for arbitrary rings R and proved in exactly the same way.

Proposition 5.4 If G is FP over R then $\operatorname{cd}_R G = \max\{n \,:\, H^n(G, RG) \neq 0\}$.

Proof of Lemma 5.3.
1. Since G is $\mathcal{O}_{\mathcal{F}} \operatorname{FP}$, WH is FP_∞ for all finite H (Corollary 2.35) and we may apply [Bie81, Corollary 3.6] to get a short exact sequence

$$0 \to H^q(WH, \mathbb{Z}[WH]) \otimes_{\mathbb{Z}} \mathbb{Q} \to H^q(WH, \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}[WH])$$
$$\to \operatorname{Tor}_1^{\mathbb{Z}}(H^{q+1}(WH, \mathbb{Z}[WH]), \mathbb{Q}) \to 0$$

$H^{q+1}(WH, \mathbb{Z}[WH])$ is $\mathbb{Z}$-flat for all q giving an isomorphism

$$H^q(WH, \mathbb{Z}[WH]) \otimes_{\mathbb{Z}} \mathbb{Q} \cong H^q(WH, \mathbb{Q}[WH])$$

Proposition 5.4 shows $n_H = \operatorname{cd}_{\mathbb{Q}} WH$. Finally, $\operatorname{cd}_{\mathbb{Q}} G \leq \mathcal{O}_{\mathcal{F}} \operatorname{cd}_{\mathbb{Z}} G$ for all groups G [BLN01, Theorem 2], so $n_{\text{id}} \leq n$.

2. If G is R-torsion free then for any finite subgroup H, $\operatorname{cd}_R N_G H \leq \operatorname{cd}_R G \leq \mathcal{O}_{\mathcal{F}} \operatorname{cd}_R G$ and $N_G H$ is FP_∞ over R by Corollary 2.35. Since

$$H^i(N_G H, R[N_G H]) \cong H^i(WH, R[WH])$$

Proposition 5.4 shows $n_H = \operatorname{cd}_R N_G H = \operatorname{cd}_R WH$. Finally, $n_{\text{id}} \leq n$ because $\operatorname{cd}_R G \leq \mathcal{O}_{\mathcal{F}} \operatorname{cd}_R G$ (Lemma 2.22).

□

Question 5.5. Is it always true that $n = n_{\text{id}}$?

Lemma 5.6 If G is Bredon duality of dimension n over $\mathbb{Z}$ then G is Bredon duality of dimension n over any ring R.

Proof. Since G is $\mathcal{O}_{\mathcal{F}}\,\text{FP}$ over $\mathbb{Z}$, G is $\mathcal{O}_{\mathcal{F}}\,\text{FP}$ over R. As in the proof of part (1) of the previous lemma there is an isomorphism for any finite subgroup H,

$$H^q(WH, \mathbb{Z}[WH]) \otimes_{\mathbb{Z}} R \cong H^q(WH, R[WH])$$

Observing that if an Abelian group M is $\mathbb{Z}$-flat then $M \otimes_{\mathbb{Z}} R$ is R-flat completes the proof.

□

5.1 Examples

In this section we provide several sources of examples of Bredon duality groups, showing that Bredon duality is not too rare a property.

5.1.1 Smooth Actions on Manifolds

If G admits a cocompact n-dimensional manifold model M for $\text{E}_{\mathcal{F}in} G$ such that M^H is a submanifold then, using [Bro94, Ex.4 p209] and Poincaré-duality [Hat02, Theorem 3.35], for any finite subgroup H,

$$H^i(WH, \mathbb{Z}[WH]) = \begin{cases} \mathbb{Z} & \text{if } i = \dim M^H \\ 0 & \text{else} \end{cases}$$

Making G into a Bredon-Poincaré duality group over $\mathbb{Z}$ and thus also over R. Note that we don't need M to be a model for $\text{E}_{\mathcal{F}in} G$ to get the condition on cohomology, only that M^H is a submanifold and the action of WH on M^H is proper and cocompact. However, the condition that M be a cocompact model for $\text{E}_{\mathcal{F}in} G$ does give the required $\mathcal{O}_{\mathcal{F}}\,\text{FP}$ condition. The following Lemma guarantees that M^H is a submanifold of M:

Lemma 5.7 [Dav08, 10.1 p.177] If G is a discrete group acting properly and locally linearly on a manifold M then the fixed points subsets of finite subgroups of G are locally flat submanifolds of M.

Locally linear is a technical condition, the definition of which can be found in [Dav08, Definition 10.1.1], for our purposes it is enough to know that if M

is a smooth manifold and G acts by diffeomorphisms then the action is locally linear. The locally linear condition is necessary - in [DL03] examples are given of virtually torsion-free groups acting as a discrete cocompact group of isometries of a CAT(0) manifold which are not Bredon duality, ie. the examples are of virtual-Poincaré duality groups which are not Bredon duality.

We can generalise Wall's conjecture, first posed in [Wal79], which asks if every finitely presented Poincaré duality group over $\mathbb{Z}$ admits a manifold model for BG.

Question 5.8. Do all finitely presented Bredon-Poincaré duality groups over $\mathbb{Z}$ admit cocompact manifold models M for $\mathrm{E}_{\mathcal{F}in}G$, where for each finite subgroup H the fixed point set M^H is a submanifold.

Example 5.9. Let p be a prime and let G be the wreath product

$$G = \mathbb{Z} \wr C_p = \left(\bigoplus_{\mathbb{Z}_p} \mathbb{Z} \right) \rtimes C_p$$

Where C_p denotes the cyclic group of order p. G acts properly and by diffeomorphisms on $\mathbb{R}^p$: The copies of $\mathbb{Z}$ act by translation along the axes, and the C_p permutes the axes. The action is cocompact with fundamental domain the quotient of the p-torus by the action of C_p. The finite subgroup C_p is a representative of the only conjugacy class of finite subgroups in G, and has fixed point set the line $\{(\lambda, \cdots, \lambda) \,:\, \lambda \in \mathbb{R}\}$. If $z = (z_1, \ldots, z_p) \in \mathbb{Z}^p$ then the fixed point set of $(C_p)^z$ is the line $\{(\lambda + z_1, \ldots, \lambda + z_p) \,:\, \lambda \in \mathbb{R}\}$.

Hence $\mathbb{R}^p$ is a model for $\mathrm{E}_{\mathcal{F}in}G$ and, invoking Lemma 5.7, G is a Bredon Poincaré duality group of dimension p with $\mathcal{V} = \{1\}$.

Example 5.10. Fixing positive integers $m \leq n$, if $G = \mathbb{Z}^n \rtimes C_2$ where C_2, the cyclic group of order 2, acts as the antipodal map on $\mathbb{Z}^{n-m} \leq \mathbb{Z}^n$ then

$$N_G C_2 = C_G C_2 = \{g \in G \,:\, gz = zg\}$$

But this is exactly the fixed points of the action of C_2 on G, hence $N_G C_2 = \mathbb{Z}^m \rtimes C_2$ and

$$H^i(N_G C_2, R[N_G C_2]) \cong \begin{cases} R & \text{if } i = m \\ 0 & \text{else.} \end{cases}$$

G embeds as a discrete subgroup of $\mathrm{Isom}(\mathbb{R}^n) = \mathbb{R}^n \rtimes GL_n(\mathbb{R})$ and acts properly and cocompactly on $\mathbb{R}^n$. It follows that G is $\mathcal{O}_\mathcal{F}\,\mathrm{FP}$ and $\mathcal{O}_\mathcal{F} \operatorname{cd} G = n$ so G is Bredon-Poincaré duality of dimension n over any ring R with $\mathcal{V} = \{m\}$.

Example 5.11. Similarly to the previous example we can take

$$G = \mathbb{Z}^n \rtimes \bigoplus_{i=1}^{n} C_2$$

Where the j^{th} copy of C_2 acts antipodally on the j^{th} copy of $\mathbb{Z}$ in $\mathbb{Z}^n$. Note that G is isomorphic to $(D_\infty)^n$ where D_∞ denotes the infinite dihedral group. As before G embeds as a discrete subgroup of $\mathrm{Isom}(\mathbb{R}^n) = \mathbb{R}^n \rtimes GL_n(\mathbb{R})$ and acts properly and cocompactly on $\mathbb{R}^n$. Thus G is $\mathcal{O}_\mathcal{F}\,\mathrm{FP}$ and $\mathcal{O}_\mathcal{F} \operatorname{cd} G = n$, so G is Bredon-Poincaré duality of dimension n over any ring R with $\mathcal{V}(G) = \{0, \ldots, n\}$.

More generally, we could take a subgroup $\bigoplus_{i=1}^m C_2 \longrightarrow \bigoplus_{i=1}^n C_2$ and form the semi-direct product of $\mathbb{Z}^n$ with this subgroup. Although this gives us a range of possible values for $\mathcal{V}(G)$ it is impossible to produce a full range of values. Consider the case $m = 2$, so we have a group

$$G = \mathbb{Z}^n \rtimes (A \times B)$$

Where $A \cong B \cong C_2$, and both A and B act either trivially or antipodally on each coordinate of $\mathbb{Z}^n$. We can describe the normaliser $N_G A$ by an element $(a_1, \ldots, a_n) \in \{0, 1\}^n$, so

$$N_G A = \left(\bigoplus_{i=1}^n \left\{ \begin{array}{ll} \mathbb{Z} & \text{if } a_i = 1 \\ 0 & \text{else.} \end{array} \right\} \right) \rtimes (A \times B)$$

Similarly we can descibe $N_G B$ by an element $(b_1, \ldots, b_n) \in \{0, 1\}^n$. One calculates that the normaliser $N_G(A \times B)$ is described by the element

$$(a_1, \ldots, a_n) \wedge (b_1, \ldots, b_n)$$

Where $\wedge$ denotes the boolean AND function.

If C denotes the subgroup of $A \times B$ generated by the element $(1, 1)$ then the normaliser of $N_G C$ is described by the element

$$\neg\big((a_1, \ldots, a_n) \oplus (b_1, \ldots, b_n)\big)$$

Where $\oplus$ denotes the boolean XOR function, and $\neg$ the unary negation operator.

Now, using the above it can be shown that, for example, a Bredon Poincaré duality group of dimension 4 with the form

$$G = \mathbb{Z}^4 \rtimes \bigoplus_{i=1}^m C_2$$

cannot have $\mathcal{V}(G) = \{1, 3\}$. Assume that such a G exists, clearly $m \geq 2$, let A and B denote two of the C_2 summands of $\oplus_{1=1}^m C_2$. Without loss of generality we can assume that A and B don't have the same action on $\mathbb{Z}^3$. If $n_A = n_B = 1$ then by the description of the normaliser of $A \times B$ above, $n_{A\times B} = 0$, a contradiction. If $n_A = n_B = 3$ then in order for A and B not to have the same action on $\mathbb{Z}^3$, we must have (up to some reordering of the coordinates)

$$(a_1, \ldots, a_4) = (1, 1, 1, 0)$$
$$(b_1, \ldots, b_4) = (0, 1, 1, 1)$$

So $n_{A\times B} = 2$, a contradiction. Finally, if $n_A = 1$ and $n_B = 3$ then let C be the subgroup of $A \times B$ generated by $(1, 1)$. There are two possibilities, up to reordering of the coordinates, either

$$(a_1, \ldots, a_4) = (1, 1, 1, 0)$$
$$(b_1, \ldots, b_4) = (1, 0, 0, 0)$$

or

$$(a_1, \ldots, a_4) = (1, 1, 1, 0)$$
$$(b_1, \ldots, b_4) = (0, 0, 0, 1)$$

In the first case, $n_C = 2$, and in the second case $n_{A\times B} = 0$, both contradictions.

Example 5.12. In [FW08, Theorem 6.1], Farb and Weinberger construct a group acting properly cocompactly and by diffeomorphisms on $\mathbb{R}^n$ for some n - and thus is a Bredon Poincaré duality group. However the group constructed is not virtually torsion-free.

Remark 5.13. Restrictions on the dimensions of the fixed point sets.

Suppose G is a group acting smoothly on an m-dimensional manifold M, and suppose furthermore that G contains a finite cyclic subgroup C_p fixing a point $x \in M$. There is an induced linear action of C_p on the tangent space $T_xM \cong \mathbb{R}^m$, equivalently a representation of C_p into the orthogonal group $O(m)$. We can use this to give some small restrictions on the possible dimensions of the submanifold M^{C_p}, and hence on the values of n_{C_p}.

A representation of C_p in $O(m)$ is simply a matrix M with $M^p = 1$. Using the Jordan-Chevalley decomposition we see that M is semi-simple, so viewing M as a matrix over $\mathbb{C}$ it is diagonalisable. However, since $M^p = 1$ and the characteristic polynomial has coefficients in $\mathbb{R}$, all the eigenvalues come in pairs ω, ω^{-1}, where ω is a p^{th} root of unity. Thus M is conjugate via complex matrices to

$$\begin{pmatrix} \omega_1 & & & & \\ & \omega_1^{-1} & & & \\ & & \ddots & & \\ & & & \omega_{\frac{m}{2}} & \\ & & & & \omega_{\frac{m}{2}}^{-1} \end{pmatrix} \text{ or } \begin{pmatrix} \omega_1 & & & & & \\ & \omega_1^{-1} & & & & \\ & & \ddots & & & \\ & & & \omega_{\frac{m-1}{2}} & & \\ & & & & \omega_{\frac{m-1}{2}}^{-1} & \\ & & & & & \pm 1 \end{pmatrix}$$

Depending on whether m is even or odd. The blank space in the matrices should be filled with zeros. Note that the ± 1 term can only be a -1 if $p = 2$. The matrix

$$\begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}$$

is conjugate via complex matrices to

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Thus M is conjugate, via complex matrices, to $R_{\theta_1} \oplus \cdots \oplus R_{\theta_{m/2}}$ or $R_{\theta_1} \oplus \cdots \oplus R_{\theta_{(m-1)/2}} \oplus (\pm 1)$, and by [Zha11, 5.11], they are conjugate via real matrices as well. Hence the fixed point sets are the same. Noting that the rotation matrix R_θ fixes only the origin when $\theta \neq 0$, we conclude that for $p \neq 2$, the fixed point set M^{C_p} must be even dimensional if m is even, and odd dimensional otherwise.

Consider the case that G is a Bredon-Poincaré duality group, arising from a smooth cocompact action on an m-dimensional manifold M, and C_p for $p \neq 2$ is some finite subgroup of G. Then n_{C_p} is exactly the dimension of the submanifold M^{C_p}, and by the discussion above n_{C_p} is odd dimensional if m is odd dimensional, even dimensional otherwise. As demonstrated by Example 5.10, there are no restrictions when $p = 2$.

5.1.2 One Relator Groups

Let G be an FP_2 torsion-free group of cohomological dimension 2 which doesn't split as a free product, this is equivalent to asking that $H^1(G, \mathbb{Z}G) = 0$ ([Bie81, Theorem 7.1], see also [Swa69]). We borrow an argument of Bieri and Eckmann in [BE73, 5.2] to prove that $H^2(G, \mathbb{Z}G)$ is a flat $\mathbb{Z}$-module and hence G is a duality group. Consider the short exact sequence of $\mathbb{Z}G$ modules

$$0 \longrightarrow \mathbb{Z}G \xrightarrow{\times p} \mathbb{Z}G \longrightarrow \mathbb{F}_p G \longrightarrow 0$$

This yields a long exact sequence

$$\cdots \longrightarrow H^1(G, \mathbb{F}_p G) \longrightarrow H^2(G, \mathbb{Z}G) \xrightarrow{\times p} H^2(G, \mathbb{Z}G) \longrightarrow \cdots$$

By the universal coefficient theorem,

$$H^1(G, \mathbb{F}_p G) = H^1(G, \mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{F}_p = 0$$

Hence the map $H^2(G, \mathbb{Z}G) \xrightarrow{\times p} H^2(G, \mathbb{Z}G)$ must have zero kernel for all p, in other words $H^2(G, \mathbb{Z}G)$ is torsion-free. But the torsion-free $\mathbb{Z}$-modules are exactly the flat $\mathbb{Z}$-modules. Thus G is duality.

Let G be a one-relator group (see [LS01, §5] for background on these groups), G has the following properties:

1. G is $\mathcal{O}_{\mathcal{F}}\,\mathrm{FP}$ and $\mathcal{O}_{\mathcal{F}}\,\mathrm{cd}_{\mathbb{Z}}\,G = 2$, since it has a cocompact 2-dimensional classifying space for proper actions [Lüc03, 4.12].
2. G contains a torsion-free subgroup Q of finite index [FKS72].

If $\mathrm{cd}_{\mathbb{Z}}\,Q \leq 1$ then Q is either finite or a finitely generated free group and G is either finite or virtually finitely generated-free. Thus G is Bredon duality over $\mathbb{Z}$ by 5.24, 5.26, and 5.25. Assume therefore that $\mathrm{cd}_{\mathbb{Z}}\,Q = 2$. Being finite index in G, Q is also FP_2 and $H^1(Q, \mathbb{Z}Q) = H^1(G, \mathbb{Z}G) = 0$, thus by the above paragraph Q is a duality group and G is virtual duality.

Every finite subgroup of G is subconjugated to a finite cyclic self-normalising subgroup C of G [LS01, 5.17,5.19], and furthermore the normaliser of any finite subgroup is subconjugate to C - if K is a non-trivial subgroup of C and $n \in N_G K$ then $n^{-1}Cn \cap C \neq 1$ and [LS01, 5.19] implies that $n \in C$. For an arbitrary non-trivial finite subgroup K', since K' is conjugate to some $K \leq C$, the normaliser $N_G K'$ is conjugate to $N_G K \leq C$.

Since the normaliser of any non-trivial finite subgroup F is finite,

$$H^i(N_G F, \mathbb{Z}[N_G F]) = \begin{cases} 0 & \text{if } i > 0, \\ \mathbb{Z} & \text{if } i = 0. \end{cases}$$

Hence G is Bredon duality of dimension 2.

Proposition 5.14 If G is a one relator group then either

1. G is finite, and hence Bredon-Poincaré duality of dimension 0 over any ring R.
2. G is virtually-free, and hence Bredon duality of dimension 1 over any ring R.
3. G is none of the above, but splits as a finite graph of groups with finite edge groups, and virtually duality vertex groups.
4. G is Bredon duality, and virtually duality, of dimension 2 over any ring R, with $\mathcal{V}(G) = \{0\}$.

Proof. It remains to show that if G is a one relator group with $H^1(G, \mathbb{Z}G) \neq 0$, then we are in situation (3) above. Since G is $\mathcal{O}_\mathcal{F}\,\mathrm{FP}_0$, G has bounded orders of finite subgroups and by a result of Linnell, G is accessible - in other words G admits a decomposition as the fundamental group of a finite graph of groups with finite edge groups and vertex groups G_v satisfying $H^1(G, \mathbb{Z}G) = 0$ [Lin83]. These vertex groups are subgroups of virtually torsion-free groups so in particular virtually torsion-free with $\mathcal{O}_\mathcal{F}\,\mathrm{cd}_\mathbb{Z}\, G \leq 2$. Lemma 5.15 below gives that the vertex groups are FP_2. Hence by the discussion at the beginning of this section, these edge groups are virtually duality. □

Lemma 5.15 Let G be a group which splits as a finite graph of groups with finite edge groups G_e, indexed by E, and vertex groups G_v, indexed by V. Then if G is FP_2, so are the vertex groups G_v.

Proof. Fix a vertex group G_v. Let M_λ, for $\lambda \in \Lambda$, be a directed system of $\mathbb{Z}G_v$ modules with $\varinjlim M_\lambda = 0$. To use the Bieri-Eckmann criterion [Bie81, Theorem 1.3], we must show that $\varinjlim H^i(G_v, M_\lambda) = 0$ for $i = 1, 2$.

The Mayer-Vietoris sequence associated to the graph of groups is

$$\cdots \longrightarrow H^i(G, -) \longrightarrow \bigoplus_{v \in V} H^i(G_v, -) \longrightarrow \bigoplus_{e \in E} H^i(G_e, -) \longrightarrow \cdots$$

Now $\varinjlim M_\lambda = 0$, so $\varinjlim \mathrm{Ind}_{\mathbb{Z}G_v}^{\mathbb{Z}G} M_\lambda = 0$ as well. Evaluating the Mayer-Vietoris sequence at $\mathrm{Ind}_{\mathbb{Z}G_v}^{\mathbb{Z}G} M_\lambda$, taking the limit, and using the Bieri-Eckmann criterion, implies

$$\varinjlim_{\Lambda} \bigoplus_{v \in V} H^i(G_v, \mathrm{Ind}_{\mathbb{Z}G_v}^{\mathbb{Z}G} M_\lambda) = 0$$

In particular $\varinjlim H^i(G_v, \mathrm{Ind}_{\mathbb{Z}G_v}^{\mathbb{Z}G} M_\lambda) = 0$, and because M_λ is a direct summand of $\mathrm{Ind}_{\mathbb{Z}G_v}^{\mathbb{Z}G} M_\lambda$ [Bro94, VII.5.1], this implies $\varinjlim H^i(G_v, M_\lambda) = 0$. □

Question 5.16. Are the groups in (3) of the Proposition also Bredon duality groups.

5.1.3 Discrete Subgroups of Lie Groups

If L is a Lie group with finitely many path components, K a maximal compact subgroup and G a discrete subgroup then L/K is a model for $\mathrm{E}_{\mathcal{F}in}G$. The space L/K is a manifold and the action of G on L/K is smooth so the fixed point subsets of finite groups are submanifolds of L/K, using Lemma 5.7. If we assume that the action is cocompact then G is seen to be of type $\mathcal{O}_\mathcal{F}\,\mathrm{FP}$, $\mathcal{O}_\mathcal{F}\,\mathrm{cd}\, G = \dim L/K$ and G is a Bredon duality group. See [Lüc03, Theorem 5.24] for a statement of these results.

Example 5.17. In [Rag84][Rag95], examples of cocompact lattices in finite covers of the Lie group $\mathrm{Spin}(2, n)$ are given which are not virtually torsion-free.

5.1.4 Soluble Groups

The reader is referred to [Rob96, §5][Rot95, §5] for background material on soluble groups. In [Kro86], Kropholler proved that for a soluble group G, the following conditions are equivalent:

1. $\operatorname{cd} G = \operatorname{hd} G < \infty$.
2. G is FP.
3. G is duality.

Additionally, if one of the above holds, G is Poincaré Duality if and only if G is polycyclic. Combining this with [Kro93], where it is shown that soluble groups of type FP_∞ are virtually of type FP, gives.

Theorem 5.18 [Kro86][Kro93] The following conditions on a virtually-soluble group G are equivalent:

1. G is FP_∞.
2. G is virtually duality.
3. G is virtually torsion-free and $\operatorname{vcd} G = hG < \infty$.

Additionally, if one of the above holds then G is virtually Poincaré duality if and only if G is virtually-polycyclic.

If G is virtually soluble and Bredon duality, then G is $\mathcal{O}_\mathcal{F}\,\mathrm{FP}_\infty$, hence also FP_∞ and virtually duality. Conversely, given a virtually soluble duality group G, [MPN10], gives that G is type $\mathcal{O}_\mathcal{F}\,\mathrm{FP}$ and $\mathcal{O}_\mathcal{F} \operatorname{cd} G = hG < \infty$. To see that G is Bredon duality we must check the cohomology condition on the Weyl groups. Since G is $\mathcal{O}_\mathcal{F}\,\mathrm{FP}$, the Weyl groups $N_G F$ of any finite subgroup F of G are FP_∞. Subgroups of virtually-soluble groups are virtually-soluble [Rob96, 5.1.1], so the normalisers $N_G F$ are virtually-soluble FP_∞ and hence virtually duality by Theorem 5.18 above, and so satisfy the required condition on cohomology. Hence G is Bredon duality.

If G is virtually soluble Poincaré-duality then G is virtually-polycyclic. Subgroups of virtually-polycyclic groups are virtually-polycyclic [Rob96, p.52], so $N_G F$ is polycyclic FP_∞ for all finite subgroups F and

$$H^{n_F}(N_G F, \mathbb{Z}[N_G F]) = \mathbb{Z}$$

Thus G is Bredon-Poincaré duality. We have arrived at the following restatement of [MP13, Example 5.6]:

Proposition 5.19 We can add the following equivalent condition to Theorem 5.18:

4. G is Bredon duality.

Additionally, if G is Bredon duality then G is virtually Poincaré duality if and only if G is virtually-polycyclic if and only if G is Bredon-Poincaré duality.

5.1.5 Elementary Amenable Groups

If G is an elementary amenable group FP_∞ group, [HL92] provides a decomposition of G as a locally-finite by virtually-soluble group. Since G is FP_∞ it has a bound on the orders of its finite subgroups [Kro93] and thus G is finite-by-virtually soluble. Moreover, [KMPN09, p.3] yields that G has $\mathcal{O}_\mathcal{F} \operatorname{cd}_\mathbb{Z} G =$

$hG < \infty$ and combining this with the result of [Sch78, Theorem 6] that the class of virtually-soluble groups of finite cohomological dimension is extension closed implies G is in fact virtually soluble.

Thus any elementary amenable FP_∞ group is virtually soluble FP_∞, in particular Bredon duality over $\mathbb{Z}$ of dimension hG. The converse, that every elementary amenable Bredon duality group is FP_∞ is obvious.

The above Proposition could be viewed as adding an additional equivalent condition to [KMPN09, Theorem 1.1], so that it now reads:

Theorem 5.20 The following conditions on an elementary amenable group G are equivalent:

1. G has cocompact classifying space for proper actions, is $\mathcal{O}_\mathcal{F}\,\mathrm{FL}$, $\mathcal{O}_\mathcal{F}\,\mathrm{FP}$ or $\mathcal{O}_\mathcal{F}\,\mathrm{FP}_\infty$.
2. G is virtually-F, virtually-FL, virtually-FP or FP_∞.
3. G is polycyclic-by-finite or G has a normal subgroup K such that G/K is Euclidean Crystallographic and for each subgroup L with $K \le L$ and L/K finite, there is a finitely generated virtually nilpotent subgroup $B = B(L)$ of L and an element $t = t(L)$ such that $t^{-1}Bt \le B$ and $L = B*_{B,t}$ is a strictly ascending HNN extension with base B and stable letter t.
4. G is virtually-duality or Bredon duality.

Additionally, if one of the above conditions is satisfied then G is Bredon-Poincaré-duality if and only if G is virtually-polycyclic if and only if G is virtually Poincaré duality.

The above theorem implies that if G is elementary amenable $\mathcal{O}_\mathcal{F}\,\mathrm{FP}_\infty$ then the condition $H^n(G, \mathbb{Z}G) \cong \mathbb{Z}$ implies that G is virtually Poincaré duality and hence Bredon-Poincaré duality, so for all finite subgroups $H^{n_F}(N_GF, \mathbb{Z}N_GF) \cong \mathbb{Z}$. A natural question is whether

$$H^{n_F}(N_GF, \mathbb{Z}[N_GF]) = \mathbb{Z}$$

can ever occur for an elementary amenable, or indeed a soluble Bredon-duality, but not Bredon-Poincaré-duality group. An example of this behaviour is given below.

Example 5.21. We construct a finite index extension of the Baumslag-Solitar group $BS(1,p)$, for p a prime.

$$BS(1,p) = \langle x, y \,:\, y^{-1}xy = x^p \rangle$$

This has a normal series

$$1 \trianglelefteq \langle x \rangle \trianglelefteq \langle\langle x \rangle\rangle \trianglelefteq BS(1,p)$$

Whose quotients are $\langle x \rangle/1 \cong \mathbb{Z}$, $\langle\langle x \rangle\rangle/\langle x \rangle \cong C_{p^\infty}$ and $BS(1,p)/\langle\langle x \rangle\rangle \cong \mathbb{Z}$. Clearly $BS(1,p)$ is finitely generated torsion-free soluble with $hBS(1,p) = 2$, but not polycyclic, since C_{p^∞} does not have max, thus $BS(1,p)$ is not Poincaré-duality. Also since $BS(1,p)$ is an HNN extension of $\langle x \rangle \cong \mathbb{Z}$ it has cohomological dimension 2 [Bie81, Proposition 6.12] and thus $\mathrm{cd}\, BS(1,p) = hBS(1,p)$. By Theorem 5.18, since $BS(1,p)$ is torsion-free, $BS(1,p)$ is a duality group.

Recall that elements of $BS(1,p)$ can be put in a normal form: $y^i x^k y^{-j}$ where $i, j \geq 0$ and if $i, j > 0$ then $n \nmid k$. Consider the automorphism φ of $BS(1,p)$, sending $x \mapsto x^{-1}$ and $y \mapsto y$, an automorphism since it is its own inverse and because the relation $y^{-1}xy = x^p$ in $BS(1,p)$ implies the relation $y^{-1}x^{-1}y = x^{-p}$. Let $y^i x^k, y^{-j}$ be an element in normal form.

$$\varphi : y^i x^k y^{-j} \mapsto y^i x^{-k} y^{-j}$$

So the only fixed points of φ are in the subgroup $\langle y \rangle \cong \mathbb{Z}$. Form the extension

$$1 \longrightarrow BS(1,p) \longrightarrow G \longrightarrow C_2 \longrightarrow 1$$

Where C_2 acts by the automorphism φ. The property of being soluble is extension closed [Rob96, 5.1.1], so G is soluble virtual duality and Bredon duality by Proposition 5.19. The normaliser

$$N_G C_2 = C_G C_2 = \{g \in G \, : \, gz = zg \text{ for the generator } z \in C_2\}$$

is the points in G fixed by φ, so $C_G C_2 \cong \mathbb{Z}$. By a standard argument $C_G C_2$ is finite index in $N_G C_2$ and thus $N_G C_2$ is virtually-$\mathbb{Z}$ and $H^1(N_G C_2, \mathbb{Z}[N_G C_2]) \cong \mathbb{Z}$. However since $BS(1,p)$ is not Poincaré duality and is finite index in G,

$$H^2(G, \mathbb{Z}G) \cong H^n(BS(1,p), \mathbb{Z}[BS(1,p)]) \cong \bigoplus_{\aleph_0} \mathbb{Z}$$

Remark 5.22. Restrictions on n_H

We can use Remark 5.13 and a Theorem, proved independently by Baues [Bau04] and Dekimpe [Dek03], that any virtually polycyclic group G can be realised as a NIL affine crystallographic group, to get restrictions on the values of n_H. The theorem states that G acts properly and cocompactly on a simply connected nilpotent Lie group of dimension hG, so by Remark 5.13, if C_p is a cyclic subgroup of G with p prime and not equal to 2, n_{C_p} is odd if hG is odd and n_{C_p} is even if hG is even.

Question 5.23. Do we have restrictions like the above when G is Bredon-Poincaré duality, but not necessarily elementary amenable?

5.2 Low Dimensions

This section is devoted to the study of Bredon duality, and Bredon-Poincaré duality, groups of low dimension. We completely classify those of dimension 0 in Lemma 5.24. We partially classify those of dimension 1 - see Propositions 5.25 and 5.28, and Question 5.27. There is a discussion of the dimension 2 case.

Recall [Bie81, Proposition 9.17(a)], that a group G is duality of dimension 0 over R if and only if G is finite and the order of G is invertible in R.

Lemma 5.24 G is Bredon duality of dimension 0 over R if and only if $|G|$ is finite. Any such group is necessarily Bredon Poincaré duality. Notice that this is independent of the ring R.

Proof. If G is Bredon duality of dimension 0 then

$$H^n(G, RG) = \begin{cases} R\text{-flat} & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}$$

But, by [Geo08, 13.2.11]

$$H^0(G, RG) = \begin{cases} R & \text{if } |G| \text{ is finite} \\ 0 & \text{else.} \end{cases}$$

Hence G is finite and moreover G is Bredon-Poincaré duality.

Conversely, if G is finite then $\mathcal{O}_{\mathcal{F}} \operatorname{cd}_R G = 0$ and G is $\mathcal{O}_{\mathcal{F}} \mathrm{FP}_\infty$ over R. Finally the Weyl groups of any finite subgroup will be finite so by [Geo08, 13.2.11,13.3.1].

$$H^n(WH, R[WH]) = \begin{cases} R & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}$$

Thus G is Bredon-Poincaré duality of dimension 0. □

Recall [Bie81, Proposition 9.17(b)] - the duality groups of dimension 1 over R are exactly the groups of type FP_1 over R (equivalently finitely generated groups [Bie81, Proposition 2.1]) with $\operatorname{cd}_R G = 1$.

Proposition 5.25 If G is an R-torsion free infinite group then the following are equivalent:

1. G is Bredon duality over R, of dimension 1.
2. G is finitely generated and virtually-free.
3. G is virtually duality over R, of dimension 1.

Proof. That $2 \Rightarrow 3$ is [Bie81, Proposition 9.17(b)]. For $3 \Rightarrow 2$, let G be virtually duality over R of dimension 1, then by [Dun79] G acts properly on a tree. Since G is assumed finitely generated, by [Ant11, Theorem 3.3] G is virtually-free.

For $1 \Rightarrow 2$, if G is Bredon duality over R of dimension 1, then G is automatically finitely generated and $\mathcal{O}_{\mathcal{F}} \operatorname{cd}_R G = 1$. By Lemma 2.22 $\operatorname{cd}_R G = 1$ so, as above, by [Dun79] and [Ant11, Theorem 3.3], G is virtually-free.

For $2 \Rightarrow 1$, if G is virtually finitely generated free then G acts properly and cocompactly on a tree, so G is $\mathcal{O}_{\mathcal{F}} \mathrm{FP}$ over R with $\mathcal{O}_{\mathcal{F}} \operatorname{cd}_R G = 1$. As G is $\mathcal{O}_{\mathcal{F}} \mathrm{FP}$, for any finite subgroup K, the normaliser $N_G K$ is finitely generated. Subgroups of virtually-free groups are virtually-free, so $N_G K$ is virtually finitely generated free, in particular:

$$H^i(WK, \mathbb{Z}[WK]) = H^i(N_G K, \mathbb{Z}[N_G K]) = \begin{cases} \mathbb{Z}\text{-flat} & \text{for } i = n_K \\ 0 & \text{else.} \end{cases}$$

where $n_K = 0$ or 1. Thus G is Bredon duality over $\mathbb{Z}$ and hence also over R. □

Remark 5.26. The only place that the condition G be R-torsion-free was used was in the implication $1 \Rightarrow 2$, the problem is the condition $\mathcal{O}_{\mathcal{F}} \operatorname{cd}_R G \leq 1$ is not known to imply that G acts properly on a tree. If we take $R = \mathbb{Z}$ then $\mathcal{O}_{\mathcal{F}} \operatorname{cd}_{\mathbb{Z}} G \leq 1$ implies G acts properly on a tree by a result of Dunwoody [Dun79]. We conclude that over $\mathbb{Z}$, G is Bredon duality of dimension 1 if and only G is finitely generated virtually free, if and only if G is virtually duality of dimension 1.

Question 5.27. What characterises Bredon-duality groups of dimension 1 over R?

We don't need the R-torsion free condition to deal with dimension 1 Bredon-Poincaré duality groups over R.

Proposition 5.28 If G is an infinite group then the following are equivalent:

1. G is Bredon-Poincaré duality over R, of dimension 1.
2. G is virtually infinite cyclic.
3. G is virtually Poincaré duality over R, of dimension 1.

Proof. The equivalence follows from the fact that for G a finitely generated group, G is virtually infinite cyclic if and only if $H^1(G, RG) \cong R$ [Geo08, 13.5.5,13.5.9]. $\square$

In dimension 2, we only deal with Bredon-Poincaré duality groups over $\mathbb{Z}$.

Lemma 5.29 If G is virtually a surface group then G is Bredon Poincaré duality.

Proof. If G is a virtual surface group, G has finite index subgroup H with H the fundamental group of some closed surface. Firstly, assume $H = \pi_1(S_g)$ where S_g is the orientable surface of genus g. If $g = 0$ then S_g is the 2-sphere and G is a finite group, thus G is Bredon-Poincaré duality by Lemma 5.24. We now treat the cases $g = 1$ and $g > 1$ seperately. If $g > 0$ then by [Mis10, Lemma 4.4(b)] G is $\mathcal{O}_{\mathcal{F}}\,\mathrm{FP}$ over $\mathbb{Z}$ with $\mathcal{O}_{\mathcal{F}}\,\mathrm{cd}_{\mathbb{Z}}\, G \leq 2$. If $g > 1$ then, in the same lemma, Mislin shows that the upper half-plane is a model for $\mathrm{E}_{\mathcal{F}in}G$ with G acting by hyperbolic isometries. Giving the upper half plane the structure of a Riemannian manifold with the Poincaré metric, this action is by isometries and [Dav08, 10.1] gives that the fixed point sets are all submanifolds, hence G is Bredon-Poincaré duality of dimension 2. If $g = 1$ then by [Mis10, Lemma 4.3], G acts by affine maps on $\mathbb{R}^2$ so again $\mathbb{R}^2$ is an $\mathrm{E}_{\mathcal{F}in}G$ whose fixed point sets are submanifolds, and thus G is Bredon-Poincaré duality of dimension 2. We conclude that *orientable* virtual Poincaré duality groups of dimension 2 groups are Bredon-Poincaré duality of the same dimension.

Now we treat the non-orientable case, so $H = \pi_1(T_k)$ where T_k is a closed non-orientable surface of genus k. In particular T_k has euler characteristic $\chi(T_k) = 2 - k$. H has an index 2 subgroup H' isomorphic to the fundamental group of the closed orientable surface of euler characteristic $2\chi(S)$, thus $H' = \pi_1(S_{k-1})$ the closed orientable surface of genus $k - 1$. If $k = 1$ then $H = \mathbb{Z}/2$ and G is a finite group, thus Bredon Poincaré duality by Lemma 5.24. Assume then that $k > 1$, we are now back in the situation above where G is virtually S_g for $g > 0$ and as such G is Bredon-Poincaré duality of dimension n, by the previous part of the proof. $\square$

Proposition 5.30 The following conditions are equivalent:

1. G is virtually Poincaré duality of dimension 2 over $\mathbb{Z}$.
2. G is virtually surface.
3. G is Bredon Poincaré duality of dimension 2 over $\mathbb{Z}$.

Proof. That $1 \Leftrightarrow 2$ is [Eck87] and that $2 \Rightarrow 3$ is Lemma 5.29. The implication $3 \Rightarrow 2$ is provided by [Bow04, Theorem 0.1] which states that any FP_2 group with $H^2(G, \mathbb{Q}G) = \mathbb{Q}$ is a virtual surface group and hence a virtual Poincaré duality group. If G is Bredon Poincaré duality of dimension 2 then $H^i(G, \mathbb{Q}G) = H^i(G, \mathbb{Z}G) \otimes \mathbb{Q} = \mathbb{Q}$ and G is FP_2 by Corollary 2.35 and we may apply the aforementioned theorem. □

The above proposition doesn't extend from Poincaré duality to just duality, as demonstrated by [Sch78] where an example, based on Higmans group, is given of a Bredon duality group of dimension 2 over $\mathbb{Z}$ which is not virtual duality. This example is extension of a finite group by a virtual duality group of dimension 2. In the theorem it is proved that the group is not virtually torsion-free, that it is Bredon duality follows from Proposition 5.40.

Question 5.31. Do there exist virtual duality groups of dimension 2 which are not Bredon duality?

Davis and Leary have examples of groups which are virtual Poincaré duality groups but not Bredon duality [DL03, Theorem 2, Example 2], their example is dimension 6.

Question 5.32. Whats the situation in dimension 2 for any ring R?

5.3 Extensions

In the classical case, extensions of duality groups by duality groups are always duality [Bie81, 9.10]. In the Bredon case the situation is more complex, for example semi-direct products of torsion-free groups by finite groups may not even be $\mathcal{O}_\mathcal{F}\,\mathrm{FP}_0$ [LN03]. Davis and Leary build examples of finite index extensions of Poincaré duality groups which are not Bredon duality, although they are $\mathcal{O}_\mathcal{F}\,\mathrm{FP}_\infty$ [DL03, Theorem 2], and examples of virtual duality groups which are not of type $\mathcal{O}_\mathcal{F}\,\mathrm{FP}_\infty$ [DL03, Theorem 1]. In [FL04], Farrell and Lafont give examples of prime index extensions of δ-hyperbolic Poincaré duality groups which are not Bredon Poincaré duality. In [MP13, §5], Martinez-Perez considers p-power extensions of duality groups over fields of characteristic p, showing that if Q is a p-group and G is Poincaré duality of dimension n over a field $\mathbb{K}$ of characteristic p then then $G \rtimes Q$ is Bredon Poincaré duality of dimension n. These results do not extend from Poincaré duality groups to duality groups however [MP13, §6].

The only really tractable case is that of a direct product of two Bredon duality groups. There are also some results in this section on low dimensional extensions.

5.3.1 Direct Products

Lemma 5.33 If G and H are $\mathcal{O}_{\mathcal{F}}$ FP over R then $G \times H$ is $\mathcal{O}_{\mathcal{F}}$ FP over R and

$$\mathcal{O}_{\mathcal{F}}\,\mathrm{cd}_R\, G \times H \leq \mathcal{O}_{\mathcal{F}}\,\mathrm{cd}_R\, G + \mathcal{O}_{\mathcal{F}}\,\mathrm{cd}_R\, H$$

Proof. That $\mathcal{O}_{\mathcal{F}}\,\mathrm{cd}_R\, G \times H \leq \mathcal{O}_{\mathcal{F}}\,\mathrm{cd}_R\, G + \mathcal{O}_{\mathcal{F}}\,\mathrm{cd}_R\, H$ is a special case of [Flu10, 3.59], the proof used involves showing that given resolutions $P_*(-)$ of $\underline{R}(-)$ by $\mathcal{O}_{\mathcal{F}}\, G$ modules and $Q_*(-)$ of $\underline{R}(-)$ by $\mathcal{O}_{\mathcal{F}}\, H$ modules, the total complex of the tensor product double complex is a projective resolution of $\underline{R}(-)$ by projective $\mathcal{O}_{\mathcal{F}}(G \times H)$ modules [Flu10, 3.54]. So to prove that $G \times H$ is $\mathcal{O}_{\mathcal{F}}$ FP it is sufficient to show that if $P_*(-)$ and $Q_*(-)$ are finite type resolutions, then so is the total complex, but this follows from [Flu10, 3.49]. □

Lemma 5.34 If L is a finite subgroup of $G_1 \times G_2$ then the normaliser $N_{G_1 \times G_2} L$ is finite index in $N_{G_1} \pi_1 L \times N_{G_2} \pi_2 L$, where π_1 and π_2 are the projection maps from $G_1 \times G_2$ onto the factors G_1 and G_2.

Proof. It's straightforward to check that

$$N_{G_1 \times G_2} L \leq N_{G_1} \pi_1 L \times N_{G_2} \pi_2 L$$

To see it is finite index, observe that $N_{G_1} \pi_1 L \times N_{G_2} \pi_2 L$ acts by conjugation on $(\pi_1 L \times \pi_2 L)/L$, but this set is finite so the stabiliser of L, which is exactly $N_{G_1 \times G_2} L$, is finite index in $N_{G_1} \pi_1 L \times N_{G_2} \pi_2 L$. □

Lemma 5.35 If G_1 and G_2 are Bredon duality (resp. Bredon Poincare duality), then $G \cong G_1 \times G_2$ is Bredon duality (resp. Bredon Poincare duality). Furthermore

$$\mathcal{V}(G_1 \times G_2) = \big\{ v_1 + v_2 \,:\, v_1 \in \mathcal{V}(G_1) \cup \{n_1(G_1)\} \text{ and } v_2 \in \mathcal{V}(G_2) \cup \{n_1(G_2)\} \big\}$$

Proof. By Lemma 5.33, $G \times H$ is $\mathcal{O}_{\mathcal{F}}$ FP. If L is some finite subgroup, the normaliser $N_G L$ is finite index in $N_{G_1} \pi_1 L \times N_{G_2} \pi_2 L$ so an application of Shapiro's Lemma [Bro94, III.(6.5) p.73] gives that for all i,

$$H^i(N_G L, R[N_G L]) \cong H^n(N_{G_1} \pi_1 L \times N_{G_2} \pi_2 L, R[N_{G_1} \pi_1 L \times N_{G_2} \pi_2 L])$$

Noting the isomorphism of RG modules

$$R[N_{G_1} \pi_1 L \times N_{G_2} \pi_2 L] \cong R[N_{G_1} \pi_1 L] \otimes R[N_{G_2} \pi_2 L]$$

The Künneth formula for group cohomology (see [Bro94, p.109]) is

$$\begin{array}{c} 0 \\ \downarrow \\ \bigoplus_{i+j=k} \left(H^i(N_{G_1}\pi_1 L, R[N_{G_1}\pi_1 L]) \otimes H^j(N_{G_1}\pi_1 L, R[N_{G_1}\pi_1 L])\right) \\ \downarrow \\ H^k(G_1 \times G_2, R[N_{G_1}\pi_1 L \times N_{G_2}\pi_2 L]) \\ \downarrow \\ \bigoplus_{i+j=k+1} \mathrm{Tor}_1(H^i(G_1, R[N_{G_1}\pi_1 L]), H^j(G_2, R[N_{G_2}\pi_2 L])) \\ \downarrow \\ 0 \end{array}$$

Note that here we are using that $R[N_{G_i}\pi_i L]$ is R-free. Since $H^i(G_1, R[N_{G_1}\pi_1 L])$ is assumed R-flat the Tor_1 term is zero. Hence the central term is non-zero only when $i = n_{\pi_1 L}$ and $j = n_{\pi_2 L}$, in which case it is R-flat. If G_1 and G_2 are Bredon-Poincaré duality then the central term in this case is R. □

5.3.2 Finite-by-Duality Groups

Throughout this section, F, G and Q will denote groups in a short exact sequence

$$0 \longrightarrow F \longrightarrow G \xrightarrow{\pi} Q \longrightarrow 0$$

Where F is finite. This section builds up to the proof of Proposition 5.40 that if Q is Bredon duality of dimension n over R, then G is also.

Lemma 5.36 $H^i(G, RG) = H^i(Q, RQ)$ for all i and any ring R.

Proof. The Lyndon-Hochschild-Serre spectral sequence associated to the extension is:

$$H^p(Q, H^q(F, RG)) \underset{p}{\Rightarrow} H^{p+q}(G, RG)$$

RG is projective as a RF-module so by [Bie81, Proposition 5.3, Lemma 5.7],

$$H^q(F, RG) = H^q(F, RF) \otimes_{RF} RG = \left\{ \begin{array}{ll} R \otimes_{RF} RG = RQ & \text{if } q = 0 \\ 0 & \text{else.} \end{array} \right\}$$

The spectral sequence collapses to $H^i(G, RG) = H^i(Q, RQ)$. □

Lemma 5.37 If Q is $\mathcal{O}_{\mathcal{F}}\,\mathrm{FP}_0$, then G is $\mathcal{O}_{\mathcal{F}}\,\mathrm{FP}_0$.

Proof. Let B_i for $i = 0, \ldots, n$ be a collection of conjugacy class representatives of all finite subgroups in Q. Let $\{B_i^j\}_j$ be a collection of conjugacy class representatives of finite subgroups in G which project onto B_i. Since F is finite $\pi^{-1}(B_i)$ is finite and there are only finitely many j for each i, we claim that these B_i^j are conjugacy class representatives for all finite subgroups in G.

Let K be some finite subgroup of G, we need to check it is conjugate to some B_i^j. $A = \pi(K)$ is conjugate to B_i, let $q \in Q$ be such that $q^{-1}Aq = B_i$ and let $g \in G$ be such that $\pi(g) = q$.

$\pi(g^{-1}Kg) = q^{-1}Aq = B_i$ so $g^{-1}Kg$ is conjugate to some B_i^j and hence K is conjugate to some B_i^j. Since we have already observed that for each $i = 0, \ldots, n$ the set $\{B_i^j\}_j$ is finite, G has finitely many conjugacy classes of finite subgroups. □

Lemma 5.38 If K is a finite subgroup of G then $N_G K$ is finite index in $N_G(\pi^{-1} \circ \pi(K))$.

Proof. $N_G K$ is a subgroup of $N_G(\pi^{-1} \circ \pi(K))$ since if $g^{-1}Kg = K$ then

$$\left(\pi^{-1} \circ \pi(g)\right)\left(\pi^{-1} \circ \pi(K)\right)\left(\pi^{-1} \circ \pi(g)\right)^{-1} = \pi^{-1} \circ \pi(K)$$

But $g \in \pi^{-1} \circ \pi(g)$ so $g\left(\pi^{-1} \circ \pi(K)\right)g^{-1} = \pi^{-1} \circ \pi(K)$.

$N_G K$ is the stabiliser of the conjugation action of G on G/K so by the above can be described as the stabiliser of the action of $N_G\left(\pi^{-1} \circ \pi(K)\right)$ on G/K by conjugation. But $N_G\left(\pi^{-1} \circ \pi(K)\right)$ maps K inside $\pi^{-1} \circ \pi(K)$ so $N_G K$ is the stabiliser of $N_G\left(\pi^{-1} \circ \pi(K)\right)$ on $\pi^{-1} \circ \pi(K)/K$.

K is finite, so $\pi(K)$ is finite and since the kernel of π is finite, $\pi^{-1} \circ \pi(K)$ is finite. Hence the stabiliser must be a finite index subgroup of $N_G\left(\pi^{-1} \circ \pi(K)\right)$. □

Lemma 5.39 If L is a subgroup of Q then $N_G \pi^{-1}(L) = \pi^{-1} N_Q L$.

Proof. If $g \in N_G \pi^{-1}(L)$ then $g^{-1}\pi^{-1}(L)g = \pi^{-1}(L)$ so applying π gives $\pi(g)^{-1}L\pi(g) = L$ and thus $g \in \pi^{-1}N_Q L$.

Conversely if $g \in \pi^{-1}(N_Q L)$ then $\pi(g)^{-1}L\pi(g) = L$ so

$$\left(\pi^{-1} \circ \pi(g)\right)^{-1} \pi^{-1}(L)\left(\pi^{-1} \circ \pi(g)\right) = \pi^{-1}(L)$$

Since $g \in \pi^{-1} \circ \pi(g)$, $g^{-1}\pi^{-1}(L)g = \pi^{-1}(L)$. □

Proposition 5.40 If Q is Bredon duality of dimension n over R then G is Bredon duality of dimension n over R.

Proof. Let K be a finite subgroup of G. We combine Lemma 5.38 and Lemma 5.39 to see that $N_G K$ is finite index in $N_G \pi^{-1} \circ \pi(K) = \pi^{-1}\left(N_Q \pi(K)\right)$. Hence

$$\begin{aligned} H^i\left(W_G K, R[W_G K]\right) &\cong H^i\left(N_G K, R[N_G K]\right) \\ &\cong H^i\left(\pi^{-1}\left(N_Q \pi(K)\right), R\left[\pi^{-1}\left(N_Q \pi(K)\right)\right]\right) \\ &\cong H^i\left(N_Q \pi(K), R\left[N_Q \pi(K)\right]\right) \\ &\cong H^i\left(W_Q \pi(K), R\left[W_Q \pi(K)\right]\right) \end{aligned}$$

Where the third isomorphism follows from Lemma 5.36 and the short exact sequence

$$1 \longrightarrow F \longrightarrow \pi^{-1}\left(N_Q \pi(K)\right) \longrightarrow N_Q \pi(K) \longrightarrow 1$$

Since Q is Bredon duality of dimension n this gives the condition on the cohomology of the Weyl groups.

G is $\mathcal{O}_{\mathcal{F}}\,\mathrm{FP}_0$ by 5.37, and $\mathcal{O}_{\mathcal{F}}\,\mathrm{cd}\,G = \mathcal{O}_{\mathcal{F}}\,\mathrm{cd}\,Q = n$ by [Nuc04, 5.5]. So by Corollary 2.35, it remains to show that the Weyl groups of the finite subgroups are FP_∞. For any finite subgroup K of G, the short exact sequence above and [Bie81, Proposition 1.4] gives that $\pi^{-1}(N_Q\pi(K))$ is FP_∞. But, as discussed at the beginning of the proof, N_GK is finite index in $N_G\pi^{-1}\circ\pi(K) = \pi^{-1}(N_Q\pi(K))$, so N_GK is FP_∞ also. □

Examining the proof above it's clear that $\mathcal{V}(G) = \mathcal{V}(Q)$.

5.3.3 Low Dimensional Extensions

Proposition 5.41 If N and Q are Bredon-Poincaré duality of dimension 1 over $\mathbb{Z}$ and there is a short exact sequence

$$0 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 0$$

then G is virtually torsion-free and Bredon-Poincaré duality of dimension 2.

Proof. By Proposition 5.28, N and Q are both virtually-$\mathbb{Z}$ in particular they are FP_∞ soluble groups, so G is virtually torsion free soluble by [Sch78, Theorem 6] and FP_∞ by [Bie81, Proposition 1.4]. Proposition 5.19 completes the proof. □

Schneebeli analyzes in [Sch78] properties necessary for a class of groups $\mathcal{C}$ to have the property that being virtually poly-$\mathcal{C}$ is extension closed, or equivalently that all poly-virtually-$\mathcal{C}$ groups are virtual poly-$\mathcal{C}$.

Theorem 5.42 [Sch78, Theorem 4].

Let $\mathcal{C}$ be a class of groups, closed under finite index subgroups and containing the trivial group, and such that every element in $\mathcal{C}$ is finitely generated torsion free. If $\mathcal{C}$ has the property that given any $Q \in \mathcal{C}$ and central extension of Q by the cyclic group of order p,

$$1 \longrightarrow C_p \longrightarrow G \longrightarrow Q \longrightarrow 1$$

then G is virtually torsion-free, then an extension of the form (virtually $\mathcal{C}$)-by-(virtually-$\mathcal{C}$) is virtually ($\mathcal{C}$-by-$\mathcal{C}$).

Corollary 5.43 Any extension of a finitely generated virtually-free group by a finitely generated virtually-free group is finitely generated virtually-(free-by-free).

Proof. Let $\mathcal{C}$ be the class of finitely generated free groups, if $Q \in \mathcal{C}$ then any extension

$$1 \longrightarrow C_p \longrightarrow G \longrightarrow Q \longrightarrow 1$$

necessarily splits, and hence G is virtually torsion-free. Thus we may apply Theorem 5.42. □

Corollary 5.44 Extensions of virtual duality groups (equivalently Bredon duality groups) of dimension 1 over R by finite groups, are virtual duality groups of dimension 1 over R, for any ring R.

Corollary 5.45 An extension of a virtual duality group of dimension 1 over R by a virtual duality group of dimension 1 over R is virtual duality of dimension 2 over R, for any ring R.

Proof. By Corollary 5.43, such a group G has finite index subgroup H which is (finitely generated free)-by-(finitely generated free), H is clearly duality so G is virtual duality. □

Question 5.46. Are the groups considered in the previous corollary also duality of dimension 2?

5.4 Graphs of Groups

In the case of ordinary duality groups, an amalgamated free product of two duality groups of dimension n over a duality group of dimension $n-1$ is duality of dimension n, similarly an HNN extension of a duality group of dimension n relative to a duality groups of dimension $n-1$ group is duality of dimension n [Bie81, 9.15]. We cannot hope for such a nice result as the normalisers of finite subgroups may be badly behaved, however there are some more restrictive cases where we can get results. For instance using amalgamated free products of Bredon duality groups we will be able to build Bredon duality groups G with arbitrary $\mathcal{V}(G)$.

We need some preliminary results, showing that a graph of groups is $\mathcal{O}_{\mathcal{F}}\,\mathrm{FP}$ if all groups involved are $\mathcal{O}_{\mathcal{F}}\,\mathrm{FP}$. The following Proposition is well known over $\mathbb{Z}$, see for example [GN12, Lemma 3.2], and the proof extends with no alterations to arbitrary rings R.

Lemma 5.47 There is an exact sequence, arising from the Bass-Serre tree.

$$\cdots \longrightarrow H^i_{\mathcal{O}_{\mathcal{F}}}(G, \not -) \longrightarrow \bigoplus_{v\in V} H^i_{\mathcal{O}_{\mathcal{F}}}\left(G_v, \mathrm{Res}^{\mathcal{O}_{\mathcal{F}} G}_{\mathcal{O}_{\mathcal{F}} G_v} \not -\right)$$
$$\longrightarrow \bigoplus_{e\in E} H^i_{\mathcal{O}_{\mathcal{F}}}\left(G_e, \mathrm{Res}^{\mathcal{O}_{\mathcal{F}} G}_{\mathcal{O}_{\mathcal{F}} G_e} \not -\right) \longrightarrow \cdots$$

Proof. The resolution of $\underline{R}(-)$ by projective contravariant modules associated to the Bass-Serre tree T of the graph of groups is

$$0 \longrightarrow \bigoplus_{e\in E} R[-, G/G_e] \longrightarrow \bigoplus_{v\in V} R[-, G/G_v] \longrightarrow \underline{R}(-) \longrightarrow 0$$

Giving a long exact sequence,

$$\cdots \longrightarrow \mathrm{Ext}^i_{\mathcal{O}_{\mathcal{F}}}(\underline{R}(-), ?) \longrightarrow \bigoplus_{v\in V} \mathrm{Ext}^i_{\mathcal{O}_{\mathcal{F}}}(R[-, G/G_v], ?)$$

$$\longrightarrow \bigoplus_{e\in E} \operatorname{Ext}^i_{\mathcal{O}_\mathcal{F}}(R[-,G/G_e],?) \longrightarrow \cdots$$

However for any subgroup H, using the fact that

$$\operatorname{Ind}^{\mathcal{O}_\mathcal{F}\,G}_{\mathcal{O}_\mathcal{F}\,H} \underline{R}(-) = R[-,G/H]$$

and the adjoint isomorphism between induction and restriction (see Section 1.3),

$$\begin{aligned}\operatorname{Ext}^i_{\mathcal{O}_\mathcal{F}}(R[-,G/H],?) &\cong \operatorname{Ext}^i_{\mathcal{O}_\mathcal{F}}(\operatorname{Ind}^{\mathcal{O}_\mathcal{F}\,G}_{\mathcal{O}_\mathcal{F}\,H} \underline{R}(-),?) \\ &\cong \operatorname{Ext}^i_{\mathcal{O}_\mathcal{F}}(\underline{R}(-),\operatorname{Res}^{\mathcal{O}_\mathcal{F}\,G}_{\mathcal{O}_\mathcal{F}\,H}?) \\ &\cong H^i_{\mathcal{O}_\mathcal{F}}(H,?)\end{aligned}$$

Making this substitution in the long exact sequence completes the proof. □

Lemma 5.48 If all vertex groups G_v are of type $\mathcal{O}_\mathcal{F}\operatorname{FP}_n$ and all edge groups G_e are of type $\mathcal{O}_\mathcal{F}\operatorname{FP}_{n-1}$ over R then G is of type $\mathcal{O}_\mathcal{F}\operatorname{FP}_n$ over R.

Proof. Let $M_\lambda(-)$, for $\lambda \in \Lambda$, be a directed system of $\mathcal{O}_\mathcal{F}$G-modules with colimit zero, for any subgroup H of G the directed system $\operatorname{Res}^{\mathcal{O}_\mathcal{F}\mathrm{G}}_{\mathcal{O}_\mathcal{F}H} M_\lambda(-)$ also has colimit zero. The long exact sequence of Lemma 5.47, and the exactness of colimits gives that for all i, there is an exact sequence

$$\cdots \longrightarrow \varinjlim_{\lambda\in\Lambda} H^{i-1}_{\mathcal{O}_\mathcal{F}}(G, M_\lambda(-)) \longrightarrow \bigoplus_{v\in V} \varinjlim_{\lambda\in\Lambda} H^i_{\mathcal{O}_\mathcal{F}}\left(G_v, \operatorname{Res}^{\mathcal{O}_\mathcal{F}\,G}_{\mathcal{O}_\mathcal{F}\,G_v} M_\lambda(-)\right)$$

$$\longrightarrow \bigoplus_{e\in E} \varinjlim_{\lambda\in\Lambda} H^i_{\mathcal{O}_\mathcal{F}}\left(G_e, \operatorname{Res}^{\mathcal{O}_\mathcal{F}\,G}_{\mathcal{O}_\mathcal{F}\,G_e} M_\lambda(-)\right) \longrightarrow \cdots$$

If $i \leq n$ then by the Bieri-Eckmann criterion (Theorem 1.28), the left and right hand terms vanish, thus the central term vanishes. Another application of the Bieri-Eckmann criterion gives that G is $\mathcal{O}_\mathcal{F}\operatorname{FP}_n$. □

Lemma 5.49 If $\mathcal{O}_\mathcal{F}\operatorname{cd}_R G_v \leq n$ for all vertex groups G_v and $\mathcal{O}_\mathcal{F}\operatorname{cd}_R G_e \leq n-1$ for all edge groups G_e then $\mathcal{O}_\mathcal{F}\operatorname{cd}_R G \leq n$.

Proof. Use the long exact sequence of Lemma 5.47. □

Lemma 5.50 If there is some integer n such that for all vertex groups G_v and all edge groups G_e, $H^i(G_v, RG_v)$ is R-flat if $i = n$ and 0 otherwise and $H^i(G_e, RG_e)$ is R-flat if $i = n-1$ and 0 else, then $H^i(G, RG)$ is R-flat if $i = n$ and 0 else.

Proof. The Mayer-Vietoris sequence associated to the graph of groups is

$$\cdots \longrightarrow H^q(G,RG) \longrightarrow \bigoplus_{v\in V} H^q(G_v, RG) \longrightarrow \bigoplus_{e\in E} H^q(G_e, RG) \longrightarrow \cdots$$

$H^q(G_v, RG) = H^q(G_v, RG_v) \otimes_{RG_v} RG$ by [Bie81, Proposition 5.4] so we have

$$H^q(G, RG) = 0 \text{ for } q \neq n$$

and a short exact sequence

$$0 \longrightarrow \bigoplus_{e\in E} H^{n-1}(G_e, RG_e) \otimes_{RG_e} RG \longrightarrow H^n(G, RG) \qquad (\star)$$

$$\longrightarrow \bigoplus_{v\in V} H^n(G_v, RG_v) \otimes_{RG_v} RG \longrightarrow 0$$

Finally, extensions of flat modules by flat modules are flat (use, for example, the long exact sequence associated to Tor_*^{RG}). □

Remark 5.51. Note that, in the lemma above, if $H^n(G, RG_v) \cong R$ and $H^{n-1}(G_e, RG_e) \cong R$ for all vertex and edge groups then $H^n(G, RG)$ will not be isomorphic to R. This is immediate from the short exact sequence $(\star)$.

Lemma 5.52 Let G be the fundamental group of a graph of groups Y. If K is a subgroup of the vertex group G_v and K is not subconjugate to any edge group then $N_G K = N_{G_v} K$.

Proof. The normaliser $N_G K$ acts on the K-fixed points of the Bass-Serre tree of (G, Y), but only a single vertex is fixed by K, so necessarily $N_G K \leq G_v$. □

Example 5.53. Let S_n denote the star graph of $n+1$ vertices - a single central vertex v_0, and a single edge connecting every other vertex v_i to the central vertex. Let G be the fundamental group of the graph of groups on S_n, where the central vertex group G_0 is torsion-free duality of dimension n, the edge groups are torsion-free duality of dimension $n-1$ and the remaining vertex groups G_i are Bredon duality of dimension n with $H^n(G, RG) \neq 0$.

By Lemmas 5.48 and 5.49, G is $\mathcal{O}_{\mathcal{F}}\mathrm{FP}_n$ of dimension n, so to prove it is Bredon duality it suffices to check the cohomology of the Weyl groups of the finite subgroups. Any non-trivial finite subgroup is subconjugate to a unique vertex group G_i, and cannot be subconjugate to an edge group since they are assumed torsion-free. If K is a subgroup of G_i then by Lemma 5.52, $H^i(N_G K, R[N_G K]) \cong H^i(N_{G_i} K, R[N_{G_i} K])$ and the condition follows as G_i was assumed to be Bredon duality. Finally, for the trivial subgroup we must calculate $H^i(G, RG)$, which is Lemma 5.50.

$\mathcal{V}(G)$ is easily calculable too,

$$\mathcal{V}(G) = \mathcal{V}(G_1) \vee \cdots \vee \mathcal{V}(G_n)$$

Where $\vee$ denotes the binary "or" operation.

Specialising the above example:

Example 5.54. A Bredon duality group with prescribed $\mathcal{V}(G)$.

Let $\mathcal{V} = \{v_1, \ldots, v_t\} \subset \{0, 1, \ldots, n-1\}$ be given. We specialise the above example. Choosing $G_i = \mathbb{Z}^n \rtimes \mathbb{Z}_2$ as in Example 5.10 so that $\mathcal{V}(G_i) = v_i$, let $G_0 = \mathbb{Z}^n$, let the edge groups be $\mathbb{Z}^{n-1}$, and choose injections $\mathbb{Z}^{n-1} \to \mathbb{Z}^n$ and $\mathbb{Z}^{n-1} \to \mathbb{Z}^n \rtimes \mathbb{Z}_2$ from the edge groups into the vertex groups. Then form the graph of groups as in the previous example to get, for G the fundamental group of the graph of groups,

$$\mathcal{V}(G) = \{v_1, \ldots, v_t\}$$

Remark 5.55. Because of Remark 5.51, the groups constructed in the example above will not be Bredon Poincaré duality groups. Thus Question 5.2, asking if it is possible to construct Bredon Poincaré duality groups with prescribed $\mathcal{V}(G)$ is still open.

5.5 Questions

Collected here are some of the questions relating to duality groups mentioned throughout this section.

In Example 5.54, Bredon duality groups with arbitrary $\mathcal{V}(G)$ are constructed, but the situation is more difficult for Bredon Poincaré duality groups. Examples 5.10 and 5.11 provide more examples of possible vectors $\mathcal{V}(G)$, including $\mathcal{V}(G) = \{i\}$ for any integer i. However we are still unable to construct, for example, vectors of the form $\{i, j\}$ for arbitrary integers $i \neq j$.

Question 5.2. Is it possible to construct Bredon Poincaré duality groups with prescribed $\mathcal{V}(G)$?

Whether groups of type $\mathcal{O}_{\mathcal{F}} \operatorname{FP}_\infty$ with $\operatorname{cd}_{\mathbb{Q}} G \neq \mathcal{O}_{\mathcal{F}} \operatorname{cd} G$ exist is still unknown, but since being Bredon duality is stronger than $\mathcal{O}_{\mathcal{F}} \operatorname{FP}_\infty$ the following may be easier to answer:

Question 5.5. Is it always true that $n = n_{\mathrm{id}}$?

Wall's conjecture asks if every finitely presented Poincaré duality group over $\mathbb{Z}$ admits a manifold model for BG [Wal79].

Question 5.8. Do all finitely presented Bredon-Poincaré duality groups over $\mathbb{Z}$ admit cocompact manifold models M for $\mathrm{E}_{\mathcal{F}in} G$, where for each finite subgroup H the fixed point set M^H is a submanifold?

Question 5.23. Do we have restrictions on the possible values of n_H as in Remark 5.22 for polycyclic groups, but for arbitrary Bredon-Poincaré duality groups.

The next question is related to Question 2.27, which asks if there is a nice characterisation of the condition $\mathcal{O}_{\mathcal{F}} \operatorname{cd}_R G = 1$.

Question 5.27. What characterises Bredon-duality groups of dimension 1 over R?

We show in Proposition 5.30 that, in dimension 2, the conditions virtually Poincare duality over $\mathbb{Z}$ and Bredon poincare duality over $\mathbb{Z}$ are equivalent, and describe an example of Schneebeli of a Bredon duality group over $\mathbb{Z}$ which is not virtually torsion-free, and hence not virtually duality [Sch78].

Question 5.31. Do there exist virtual duality groups of dimension 2 which are not Bredon duality? What is the lowest dimension for which such an example can exist?

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