Finiteness Conditions in Bredon Cohomology and Centralisers in Houghton's Groups*

Nine Month Report

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*The second part of this report is essentially the article *Centralisers in Houghton's Groups*, Simon St John-Green, arXiv:1207.1597

Background

This report is split into two parts, the first part is a discussion of finiteness conditions in Bredon cohomology, this provided the motivation for my study of centralisers in Houghton's groups, the topic of the second part. The following short introduction is designed to give some background information on how finiteness conditions originated and draw parallels between these and finiteness conditions in Bredon cohomology. It will not be required for the rest of the report.

In the 1930's Hurewitz made the fundamental observation that an aspherical space X is uniquely determined, up to homotopy equivalence, by its fundamental group $\pi_1(X)$, this space is now called a *model for* $\mathrm{B}\,G$ or *Eilenberg-Mac Lane space* $K(G,1)$. From here one can use invariants of these spaces to study the groups themselves, for example, calculating the co-homology $H^*(X)$ gives the group co-homology $H^*(G)$. By the 1940's a purely algebraic definition of group co-homology was formulated, replacing the space X with a projective resolution of $\mathbb{Z}$ by $\mathbb{Z}G$-modules.

There is a simple method of constructing a model X for $\mathrm{B}\,G$ via a presentation $G = \langle S|R\rangle$: Take $\vee_S S^1$, attach 2-cells for each relation and attach n-cells for $n \geq 3$ to kill the higher homotopy groups[Hat01, p.365]. From the construction we see that a group G is finitely generated if and only if there is a model for $\mathrm{B}\,G$ with finite 1-skeleton, and finitely presented if and only if there is a model for $\mathrm{B}\,G$ with finite 2-skeleton. Generalising this, we say that a group has type F_n if it admits a model for BG with finite n-skeleton. The algebraic side to these conditions are the FP_n conditions, we say that a group has type FP_n if $\mathbb{Z}$ admits a projective resolution of $\mathbb{Z}G$-modules, finitely generated up to dimension n. G being of type F_n implies it is of type FP_n as we can take the free resolution $\mathbb{Z}G$-modules arising from the chain complex associated to the universal cover of a model for $\mathrm{B}\,G$. This universal cover is called a *model for* $\mathrm{E}\,G$, or *classifying space for free actions* and it follows from an application of Whiteheads theorem that a model X for $\mathrm{E}\,G$ is the terminal object in the homotopy category of free G-CW complexes, in less technical language - for any CW complex Y with a free G-action there is a map $Y \to X$, unique up to G-homotopy equivalence.

F_1, FP_1 and finitely generated are all equivalent conditions but the situation doesn't stay as nice for larger n. In general F_n implies FP_n and F_2 (finitely presented) with FP_n implies F_n. Examples of Bestvina and Brady show there exists groups that are FP_n but not F_2 for all n. [BB97]

We say G has *geometric dimension* $\mathrm{gd}\,G \leq n$ if there exists a model for $\mathrm{B}\,G$ with no cells in dimension $> n$. On the algebraic side, G has *co-homological dimension* $\mathrm{cd}\,G \leq n$ if there is a projective resolution of $\mathbb{Z}$ by projective $\mathbb{Z}G$ modules of length n. $\mathrm{gd}\,G = 0$ if and only if $\mathrm{cd}\,G = 0$ if and only if G is the trivial group and by a theorem of Stallings and Swan, $\mathrm{cd}\,G = 1$ if and only if $\mathrm{gd}\,G = 1$ if and only if G is a free group. [Sta68][Swa69] That $\mathrm{cd}\,G = \mathrm{gd}\,G$ in general is known as the *Eilenberg-Ganea conjecture* and has been proved for all cases except the possibility that $\mathrm{cd}\,G = 2$ and $\mathrm{gd}\,G = 3$. [EG57].

For an overview of finiteness conditions see [Bro82, Chapter VIII], [Bie76] and [Geo08, Chapter II].

Spaces which admit free G-actions can be very difficult to find, not many occur "in nature" and they may be large and unwieldy. If G is a finite group, for instance, a model for $\mathrm{E}\,G$ is necessarily infinite dimensional. Instead we might look to weaken the freeness condition, looking for spaces which admit proper actions (where the cell stabilisers are finite subgroups of G). In direct analogy to the definition of a model for $\mathrm{E}\,G$, we say X is a *model for* $\underline{\mathrm{E}}\,G$ if it is terminal in the homotopy category of G-CW complexes with finite stabilisers. There are many natural constructions of models for $\underline{\mathrm{E}}\,G$ for different classes of groups. [Lü03]. This idea can be further generalised to the study of models for $\mathrm{E}_{\mathcal{F}}\, G$, terminal objects in the homotopy category of G-CW complexes with stabilisers in the family $\mathcal{F}$ of subgroups of G.

Models for $\underline{\mathrm{E}}\, G$ and models for $\mathrm{E}_{\mathcal{VC}yc}\, G$, where $\mathcal{VC}yc$ denotes the family of virtually cyclic subgroups, have recently become of great interest because they appear on one side of the Baum-Connes and Farrell-Jones conjectures respectively [LR05]. These are deep conjectures which have far reaching consequences in mathematics. The Baum-Connes conjecture is known to imply, for example, the Novikov Conjecture, Idempotent Conjecture and Trace Conjectures [MV03, p.71] and the Farrell-Jones conjecture is related to the Bass and Kaplansky conjectures [BLR08].

The homology theory which most closely reflects the world of proper actions is Bredon co-homology, defined first by Bredon in [Bre67], and extended to arbitrary classes of groups by Lück. [Lüc89] We get parallels of F_n, FP_n, cohomological and geometric dimension here too and the first part of this report will be devoted to discussing these conditions.

1 Bredon Cohomology

Throughout this report G is a discrete group and $\mathcal{F}$ is a family of subgroups of G, closed under taking subgroups and conjugation. The *orbit category*, denoted $\mathcal{O}_{\mathcal{F}}\mathrm{G}$, is the small category whose objects are the transitive G-sets G/H for $H \in \mathcal{F}$ and whose arrows are all G-maps between them. Any G-map $G/H \to G/K$ is determined entirely by the image of the coset H in G/K, so $H \mapsto xK$ is a G-map if and only if $HxK = xK \Leftrightarrow x^{-1}Hx \leq K$. We will commonly refer to the families $\mathcal{F}in$, $\mathcal{VC}yc$ and $\mathcal{T}riv$ which denote the families of finite, virtually cyclic and the trivial family of subgroups respectively.

A right (left) $\mathcal{O}_{\mathcal{F}}\mathrm{G}$-*module*, or *Bredon module*, is a contravariant (covariant) functor from $\mathcal{O}_{\mathcal{F}}\mathrm{G}$ to the category **Ab** of Abelian groups. As such, the category **Mod**-$\mathcal{O}_{\mathcal{F}}\mathrm{G}$ ($\mathcal{O}_{\mathcal{F}}\mathrm{G}$-**Mod**) of right (left) $\mathcal{O}_{\mathcal{F}}\mathrm{G}$ modules is Abelian and exactness is defined pointwise - a short exact sequence

$$M' \longrightarrow M \longrightarrow M''$$

is exact if and only if

$$M'(G/H) \longrightarrow M(G/H) \longrightarrow M''(G/H)$$

is exact for all $H \in \mathcal{F}$ [Wei94, A.3.3]. Whenever $\mathcal{O}_{\mathcal{F}}\mathrm{G}$ modules are mentioned, with no mention of left or right, the statement applies to both left and right $\mathcal{O}_{\mathcal{F}}\mathrm{G}$-modules.

The following result plays a crucial role when dealing with free $\mathcal{O}_{\mathcal{F}}\mathrm{G}$ modules. Below, Mor refers to the morphisms in the functor category of $\mathcal{O}_{\mathcal{F}}\mathrm{G}$ modules, as such these are natural transformations between functors $\mathcal{O}_{\mathcal{F}}\mathrm{G} \to$ **Ab**. $\mathbb{Z}[-, G/H] \in$ **Mod**-$\mathcal{O}_{\mathcal{F}}\mathrm{G}$ is the functor taking $G/K \mapsto \mathbb{Z}[G/K, G/H]$.

Lemma 1.0.1 The Yoneda-type Lemma [MV03, p.9] For any $M \in$ **Mod**-$\mathcal{O}_{\mathcal{F}}\mathrm{G}$ and $H \in \mathcal{F}$ the evaluation map

$$\begin{aligned} \mathrm{Mor}(\mathbb{Z}[-, G/H], M) &\to M(G/H) \\ f &\mapsto f(\mathrm{id} : G/H \to G/H) \end{aligned}$$

is an isomorphism. Analagously, if $M \in \mathcal{O}_{\mathcal{F}}\mathrm{G}$-**Mod** and $H \in \mathcal{F}$ we have the following isomorphism

$$\begin{aligned} \mathrm{Mor}(\mathbb{Z}[G/H, -], M) &\xrightarrow{\cong} M(G/H) \\ f &\mapsto f(\mathrm{id} : G/H \to G/H) \end{aligned}$$

1.1 Projective and Free Bredon Modules

In ordinary homological algebra, free right modules are constructed as left adjoint to the forgetful functor **Mod-R** $\longrightarrow$ **Set**. In the category of Bredon modules, the forgetful functor is

$$\begin{aligned} U : \mathbf{Mod}\text{-}\mathcal{O}_{\mathcal{F}}\mathrm{G} &\to [\mathcal{F}, \mathbf{Set}] \\ U : \alpha(-) &\mapsto U_{\mathbf{Ab}} \circ \alpha(-) \end{aligned}$$

Where $U_{\mathbf{Ab}}$ is the usual forgetful functor **Ab** $\longrightarrow$ **Set**, and $\mathcal{F}$ is the category whose objects are $H \in \mathcal{F}$ and which has no arrows. We can construct explicitly the free functor F which is left adjoint to U: Given $\gamma(-) \in [\mathcal{F}, \mathbf{Set}]$, set

$$F(\gamma) = \bigoplus_{\substack{x \in \gamma(H) \\ H \in \mathcal{F}}} P_{x,H} \text{ where } P_{x,H} = \mathbb{Z}[-, G/H]$$

The adjoint condition now follows, we will write $\mathrm{Mor}_{\mathbf{C}\to\mathbf{D}}(A,B)$ for the morphisms (natural transformations) between functors $A, B : \mathbf{C} \to \mathbf{D}$, in order to emphasize the categories $\mathbf{C}$ and $\mathbf{D}$:

$$\begin{aligned}
\mathrm{Mor}_{\mathcal{O}_\mathcal{F}\mathrm{G}\to\mathbf{Ab}}(F\gamma,\alpha) &= \mathrm{Mor}_{\mathcal{O}_\mathcal{F}\mathrm{G}\to\mathbf{Ab}}\left(\bigoplus_{\substack{x\in\gamma(H)\\ H\in\mathcal{F}}} P_{x,H},\alpha\right)\\
&\cong \prod_{\substack{x\in\gamma(H)\\ H\in\mathcal{F}}} \mathrm{Mor}_{\mathcal{O}_\mathcal{F}\mathrm{G}\to\mathbf{Ab}}\left(P_{x,H},\alpha\right)\\
&\cong \prod_{\substack{x\in\gamma(H)\\ H\in\mathcal{F}}} \alpha(G/H)\\
&\cong \mathrm{Mor}_{\mathcal{F}\to\mathbf{Set}}\left(\gamma, U\alpha\right)
\end{aligned}$$

The first isomorphism is because Mor takes direct sums to direct products in any abelian category [Wei94, Ex A.1.4] and the second isomorphism is the Yoneda-type isomorphism (Lemma 1.0.1). The third isomorphism comes from the observation that since $\mathcal{F}$ is a category with no arrows, an element $N \in \mathrm{Mor}_{\mathcal{F}\to\mathbf{Set}}(\gamma_1,\gamma_2)$ is simply a collection of functions $N(H) : \gamma_1(H) \to \gamma_2(H)$, equivalently a choice of element

$$N \in \prod_{\substack{x\in\gamma_1(H)\\ H\in\mathcal{F}}} \gamma_2(G/H)$$

To summarise, free right $\mathcal{O}_\mathcal{F}$G modules are direct sums of modules of the form $\mathbb{Z}[-,G/H]$ for $H \in \mathcal{F}$. Analagously, free left $\mathcal{O}_\mathcal{F}$G modules are direct sums of modules of the form $\mathbb{Z}[G/H,-]$.

We say that a free right (left) $\mathcal{O}_\mathcal{F}$G module is finitely generated if it a direct sum of finitely many modules of the form $\mathbb{Z}[-,G/H]$ (respectively $\mathbb{Z}[G/H,-]$), and an arbitrary $\mathcal{O}_\mathcal{F}$G module M is finitely generated if it admits an epimorphism $F \twoheadrightarrow M$ where F is finitely generated free.

Projective modules are defined as in any Abelian category [Wei94, 2.2]: P is projective if for any epimorphism $g : B \to C$ and map $f : P \to C$ there exists a map $f' : P \to B$ such that $f = g \circ f'$

$$\begin{array}{ccccc}
 & & P & & \\
 & & \downarrow f & \searrow f' & \\
B & \xrightarrow{g} & C & \longrightarrow & 0
\end{array} \tag{1}$$

Proposition 1.1.1 [Wei94, 2.2.1,2.2.3] The following are equivalent for an $\mathcal{O}_\mathcal{F}$G module P:

1. P is projective.
2. P satisfies the condition of (1).
3. P is a direct summand of a free module.
4. $\mathrm{Mor}(P,-)$ is an exact functor.

Lemma 1.1.2 Free modules are projective.

Proof. Let

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be an exact sequence of right $\mathcal{O}_{\mathcal{F}}$G modules, then applying Mor($\mathbb{Z}[-, G/H], -$) gives

$$0 \longrightarrow M'(G/H) \longrightarrow M(G/H) \longrightarrow M''(G/H) \longrightarrow 0$$

by the Yoneda-type isomorphism Lemma 1.0.1, but this is exact by assumption. This completes the proof as any free module is a direct sum of modules of the form $\mathbb{Z}[-, G/H]$ for some $H \in \mathcal{F}$, Mor($-, M$) takes direct sums to direct products and direct products are exact ($\mathcal{O}_{\mathcal{F}}$G-**Mod** and **Mod**-$\mathcal{O}_{\mathcal{F}}$G are AB4* Abelian categories). [Wei94, Ex A.1.4, Ex A.4.5] □

As a corollary of the above, the categories $\mathcal{O}_{\mathcal{F}}$G-**Mod** and **Mod**-$\mathcal{O}_{\mathcal{F}}$G have enough projectives, since we can always choose a sufficiently large free module which surjects onto any $\mathcal{O}_{\mathcal{F}}$G module M. For example the module $F \circ U(M)$, where F and U are the free and forgetful functors mentioned at the beginning of this section, will always surject onto M. This implies that every $\mathcal{O}_{\mathcal{F}}$G module has a projective resolution and means the "Fundamental theorem of Homological Algebra", that any two projective resolutions are unique up to homotopy equivalence, carries over to the categories of $\mathcal{O}_{\mathcal{F}}$G modules. [Wei94, 2.2.5,2.2.5]

1.1.1 Technical Results

The rest of this section is devoted to useful technical results on free right $\mathcal{O}_{\mathcal{F}}$G modules that we require later on, these are all well known but proofs are difficult to find in the literature so I include them for completeness. All these results have equivalent formulations for left $\mathcal{O}_{\mathcal{F}}$G modules, but we will not be needing them.

Lemma 1.1.3 If Ω is G-set with stabilisers in $\mathcal{F}$ then $\mathbb{Z}[-, \Omega]$ is a free right $\mathcal{O}_{\mathcal{F}}$G-module and any free right $\mathcal{O}_{\mathcal{F}}$G-module arises in this way. Ω is G-finite if and only if $\mathbb{Z}[-, \Omega]$ is finitely generated.

Proof. Decomposing Ω into G-orbits gives $\Omega = \coprod_{i \in I} G/G_i$ as G-sets. where $G_i \in \mathcal{F}$. The first part of the Proposition now follows from the following observation:

$$\mathbb{Z}[-, \Omega] = \mathbb{Z}\left[-, \coprod_{i \in I} G/G_i\right] = \bigoplus_{i \in I} \mathbb{Z}[-, G/G_i]$$

For the converse let $\bigoplus_{i \in I} \mathbb{Z}[-, G/G_i]$ be a free $\mathcal{O}_{\mathcal{F}}$G-module with $G_i \in \mathcal{F}$ and define

$$\Omega = \coprod_{i \in I} G/G_i$$

Finally it is clear that Ω is G-finite if and only if I is finite, ie. $\mathbb{Z}[-, \Omega]$ is finitely generated. □

If Ω is a G-set with stabilisers in $\mathcal{F}$ and $\mathbb{Z}[-, \Omega]$ is a free $\mathcal{O}_{\mathcal{F}}$G-module then

$$\mathbb{Z}[G/H, \Omega] \cong \mathbb{Z}[\Omega^H]$$

since any G-map $G/H \to \Omega$ is determined by the image of H in Ω and this must lie in Ω^H. The action of G on Ω induces an action of the Weyl group $WH = N_G H/H$ on Ω^H, turning $\mathbb{Z}[\Omega^H]$ into a $\mathbb{Z}[WH]$-module.

Proposition 1.1.4 If Ω is a G-set then $\mathbb{Z}[G/H, \Omega] \neq 0$ if and only if H is subconjugated to a stabiliser of Ω.

Proof. Ω can be split up into G-orbits $\Omega = \coprod_i G/G_i$ where G_i is a stabiliser, then $\mathbb{Z}[G/H, \Omega] \neq 0$ if and only if for some i, $\mathbb{Z}[G/H, G/G_i] \neq \emptyset$ ie. there exists a G-map $G/H \longrightarrow G/G_i$. This map is determined by where it sends H and $H \mapsto xG_i$ is a G-map if and only if $hxG_i = xG_i$ for all $h \in H$ if and only if $x^{-1}Hx \leq G_i$. □

Proposition 1.1.5 For $H, K \leq G$,

$$\mathbb{Z}[G/H, G/K] = \mathbb{Z}[(G/K)^H] = \bigoplus_x \mathbb{Z}[WH/WH_{xK}]$$

as WH-modules, where x runs over a set of coset representatives of the subset of the set of N_GH-K double cosets.

$$\{x \in N_GH\backslash G/K \,:\, x^{-1}Hx \leq K\}$$

and the stabilisers are given by

$$WH_{xK} = \left(N_GH \cap xKx^{-1}\right)/H$$

Proof. $[G/H, G/K] = (G/K)^H$ is clear and xK, yK are in the same WH-orbit if there exists some $nH \in WH$ (where $n \in N_GH$) with

$$nHxK = yK \Leftrightarrow nxK = yK \Leftrightarrow (N_GH)xK = (N_GH)yK$$

Combining this with the fact that $xK \in (G/K)^H$ if and only if $x^{-1}Hx \leq K$ means there is a WH-orbit for each N_GH-K double coset N_GHxK such that $x^{-1}Hx \leq K$, ie coset representatives for

$$\{x \in N_GH\backslash G/K \,:\, x^{-1}Hx \leq K\}$$

are orbit representatives for the WH-orbits in $[G/H, G/K]$. The $N_G(H)$-stabiliser of the point $xK \in (G/K)^H$ is the set

$$\{g \in N_G(H) \,:\, gxK = xK\} = \{g \in N_G(H) \,:\, g \in xKx^{-1}\} = N_G(H) \cap xKx^{-1}$$

So the WH-stabiliser of $xK \in (G/K)^H$ is $WH_{xK} = (N_G(H) \cap xKx^{-1})/H$. □

There is a useful corollary to the above result, but before it can be stated a quick technical result is needed.

Proposition 1.1.6 If M is FP_∞ as a $\mathbb{Z}F$-module for some finite subgroup $F \leq G$, then $\text{Ind}_F^G M = \mathbb{Z}G \otimes_{\mathbb{Z}F} M$ is FP_∞ as a $\mathbb{Z}G$-module.

Proof. Firstly, here is a proof of "Shapiro's Lemma" for Tor, because I couldn't find one in the literature. Let M and N be $\mathbb{Z}G$ modules and choose a projective resolution P_* of N over $\mathbb{Z}G$, this is also a resolution over $\mathbb{Z}F$ and

$$\text{Tor}_*^F(N, M) = H_*(P_* \otimes_{\mathbb{Z}F} M) = H_*(P_* \otimes_{\mathbb{Z}G} \mathbb{Z}G \otimes_{\mathbb{Z}F} M) = \text{Tor}_*^G(N, \text{Ind}_F^G M)$$

Recall also that $\text{Tor}_*(N, M) = \text{Tor}_*(M, N)$ so this also gives us

$$\text{Tor}_*^F(M, N) = \text{Tor}_*^G(\text{Ind}_F^G M, N)$$

Let $\prod_i N_i$ be an arbitrary direct product of $\mathbb{Z}G$-modules, then

$$\begin{aligned}\text{Tor}_*^G\left(\text{Ind}_H^G M, \prod_i N_i\right) &= \text{Tor}_*^F\left(M, \prod_i N_i\right)\\ &= \prod_i \text{Tor}_*^F(M, N_i)\\ &= \prod_i \text{Tor}_*^G\left(\text{Ind}_H^G M, N_i\right)\end{aligned}$$

where the first and third equalities come from Shapiro's Lemma. This finishes the proof as $\text{Ind}_H^G M$ is FP_∞ if and only if $\text{Tor}_*^{\mathbb{Z}G}(\text{Ind}_H^G M, -)$ commutes with direct products [Bro82, Theorem VIII.4.8]. □

Corollary 1.1.7 In the situation of Proposition 1.1.5, if $H, K \in \mathcal{Fin}$ then $\mathbb{Z}[G/H, G/K]$ is a finite direct sum of projective FP_∞ WH-permutation modules with finite stabilisers. In particular $\mathbb{Z}[G/H, G/K]$ is FP_∞.

Proof. Since K is finite, the set $\{x \in N_G H \backslash G/K : x^{-1}Hx \leq K\}$ is finite and $\mathbb{Z}[WH]$ can be written as a finite direct summand

$$\mathbb{Z}[G/H, G/K] = \bigoplus_x \mathbb{Z}[WH/WH_{xK}]$$

WH_{xK} is a finite group and as such $\mathbb{Z}$ is FP_∞ as a $\mathbb{Z}[WH_{xK}]$-module.

$$\mathbb{Z}[WH/WH_{xK}] = \mathrm{Ind}_{WH_{xK}}^{WH} \mathbb{Z}$$

so we may apply Proposition 1.1.6 and deduce that $\mathbb{Z}[WH/WH_{xK}]$ is FP_∞ as a $\mathbb{Z}[G]$-module. Finally, any finite direct product of FP_∞ modules is FP_∞. □

1.2 Finiteness Conditions

1.2.1 Co-homological and Geometric Dimension

We denote by $\mathbb{Z}_\mathcal{F}$ the $\mathcal{O}_\mathcal{F}$G-module taking all objects to $\mathbb{Z}$ and all arrows to the identity map. Analagously to ordinary group co-homology we define the Bredon *co-homological dimension* of a $\mathcal{O}_\mathcal{F}$G-module M to be the shortest length of a projective resolution of M by $\mathcal{O}_\mathcal{F}$G-modules, and the co-homological dimension of a group G to be the shortest length of a projective resolution of the $\mathcal{O}_\mathcal{F}$G-module $\mathbb{Z}_\mathcal{F}$. These two integers are denoted $\mathrm{pd}_\mathcal{F} M$ and $\mathrm{cd}_\mathcal{F} G$, if $\mathcal{F} = \mathcal{Fin}$ (the family of finite subgroups) then the notation $\underline{\mathrm{cd}}\, G$ is used and if $\mathcal{F} = \mathcal{VCyc}$ (the family of virtually cyclic subgroups) then the notation $\underline{\underline{\mathrm{cd}}}\, G$ is used.

In ordinary group co-homology, a model for $\mathrm{E}\, G$ is the, unique up to homotopy equivalence, contractible free G-CW complex. Equivalently it is the terminal object in the homotopy category of free G-CW complexes. Analagously, $\underline{\mathrm{E}}\, G$ is the terminal object in the homotopy category of proper G-CW complexes. Throughout this report, all actions will be assumed to be rigid - the setwise and pointwise stabilisers of cells coincide.

Theorem 1.2.1 [Lü03, Theorem 1.9] For any family $\mathcal{F}$ of subgroups of G, there exists a model for $\mathrm{E}_\mathcal{F}\, G$ and a G-CW complex is a model for $\mathrm{E}_\mathcal{F}\, G$ if and only if for all subgroups $H \leq G$:

$$X^H \simeq \begin{cases} \mathrm{pt} & \text{if } H \in \mathcal{F} \\ \emptyset & \text{if } H \notin \mathcal{F} \end{cases}$$

See [BCH94, Appendix 1] for a description of a general construction of a model for $\mathrm{E}_\mathcal{F}\, G$.

The Bredon *geometric dimension* of a group G, denoted $\mathrm{gd}_\mathcal{F}\, G$, is defined to be the minimal dimension of a model for $\mathrm{E}_\mathcal{F}\, G$. In the case where $\mathcal{F} = \mathcal{Triv}$, the family consisting of only the trivial subgroup, a model for $\mathrm{E}_{\mathcal{Triv}}\, G$ is $\mathrm{E}\, G$, the universal cover of an Eilenberg-Mac Lane space $\mathrm{K}(G, 1)$. An n-dimensional model for $\mathrm{E}_\mathcal{F}\, G$ gives rise to a free resolution of right $\mathcal{O}_\mathcal{F}$G-modules C_* by setting $C_n(G/H) = K_n(X^H)$, where K_n denotes the ordinary chain complex of a topological space. Immediately we deduce that $\mathrm{cd}_\mathcal{F}\, G \leq \mathrm{gd}_\mathcal{F}\, G$.

A theorem of Lück and Meintrup gives an inequality in the other direction:

Theorem 1.2.2 [LM00, Theorem 0.1] $\mathrm{gd}_{\mathcal{F}} G \leq \max\{\mathrm{cd}_{\mathcal{F}} G, 3\}$

If $\mathcal{F} = \mathcal{F}in$ we denote the geometric dimension by $\underline{\mathrm{gd}}\, G$, and if $\mathcal{F} = \mathcal{VC}yc$, by $\underline{\underline{\mathrm{gd}}}\, G$. Dunwoody has shown that $\underline{\mathrm{cd}}\, G = 1$ implies that $\underline{\mathrm{gd}}\, G = 1$ [Dun79], hence $\underline{\mathrm{cd}}\, G = \underline{\mathrm{gd}}\, G$ unless $\underline{\mathrm{cd}}\, G = 2$ and $\underline{\mathrm{gd}}\, G = 3$. Brady, Leary and Nucinkis show in [BLN01] that this can indeed happen. There are also interesting results for the family of virtually cyclic subgroups, in [Flu10, p.129] it is shown that for countable torsion-free soluble groups G, $\underline{\underline{\mathrm{cd}}}\, G = 1$ implies that $\underline{\underline{\mathrm{gd}}}\, G = 1$ but it is unknown whether this is true in general. A construction due to Fluch and Leary shows there are groups with $\underline{\underline{\mathrm{cd}}}\, G = 2$ but $\underline{\underline{\mathrm{gd}}}\, G = 3$ [FL]. This is an interesting contrast to the case of ordinary group co-homology where it is still an open problem if there exist groups G with $\mathrm{cd}\, G = 2$ but $\mathrm{gd}\, G = 3$.

There are many groups for which good models for $\underline{\mathrm{E}}\, G$ are known, [Lü03] is a good reference. Many papers discuss general constructions of models for $\mathrm{E}_{\mathcal{F}}\, G$ for the families $\mathcal{F}in$, of finite groups and $\mathcal{VC}yc$, of virtually cyclic groups. [KM98] [Lü00] [LM00] [LN01b] [Vog02] [MS02] [JPL06] [LW12] [Mis09] [DP12]

In [Lü00, Section 3], there are results bounding the Bredon geometric dimension for group extensions and in [MP02] a spectral sequence for Bredon cohomology is found, and used to give similar results for the Bredon cohomological dimension of group extensions.

1.2.2 $\underline{\mathrm{FP}}_n$ Conditions

The $\underline{\mathrm{FP}}_n$-conditions are natural generalisations of the FP_n conditions of ordinary group co-homology. An $\mathcal{O}_{\mathcal{F}}$G-module M is $\underline{\mathrm{FP}}_n$ (respectively $\underline{\underline{\mathrm{FP}}}_n$) if it admits a resolution by $\mathcal{O}_{\mathcal{F}in}G$-modules (respectively $\mathcal{O}_{\mathcal{VC}yc}G$-modules), which is finitely generated in all dimensions $\leq n$. A group G is $\underline{\mathrm{FP}}_n$ (respectively $\underline{\underline{\mathrm{FP}}}_n$), if $\mathbb{Z}_{\mathcal{F}in}$ (respectively $\mathbb{Z}_{\mathcal{VC}yc}$) is $\underline{\mathrm{FP}}_n$ (respectively $\underline{\underline{\mathrm{FP}}}_n$). Later in this report, the weaker finiteness conditions quasi-$\underline{\mathrm{FP}}_n$ are defined. Versions of the Bieri-Eckmann criterion for both $\underline{\mathrm{FP}}_n$ and quasi-$\underline{\mathrm{FP}}_n$ $\mathcal{O}_{\mathcal{F}}$G-modules were proved in [MPN, Section 5] (see [Bie76, Section 1.3] for the classical case.)

The next few Lemmas detail an alternative algebraic description of the condition $\underline{\mathrm{FP}}_n$ which is easier to calculate.

Proposition 1.2.3 [KMPN09, Lemma 3.1] G is $\underline{\mathrm{FP}}_0$ if and only if G has finitely many conjugacy classes of finite subgroups.

Proof. If G is $\underline{\mathrm{FP}}_0$ then there is a finitely generated free right $\mathcal{O}_{\mathcal{F}}$G-module F and an epimorphism $F \twoheadrightarrow \mathbb{Z}_{\mathcal{F}}$, since F is free by Lemma 1.1.3 there is a G-finite G-set Ω with finite stabilisers such that $F = \mathbb{Z}[-, \Omega]$. Let G_x denote the point stabiliser of $x \in \Omega$, since $gG_xg^{-1} = G_{gx}$ for any $g \in G$, there is at most one conjugacy class for each orbit. There are only finitely many orbits so we may deduce there is only a finite set of conjugacy classes of finite subgroups of point stabilisers of Ω.

Let K be a finite subgroup of G, evaluating $\mathbb{Z}[-, \Omega]$ at G/K gives a surjection

$$\mathbb{Z}[G/K, \Omega] = \mathbb{Z}[\Omega^K] \twoheadrightarrow \mathbb{Z}$$

This implies that Ω^K is non-empty, so K stabilises a point and is a subgroup of a point stabiliser and hence a member of one of the finite set of conjugacy classes of finite subgroups of point stabilisers.

For the converse, if G has only finitely many conjugacy classes of finite subgroups then we may take $\Omega = \coprod_{H \in X} G/H$ where H runs over the set of conjugacy class representatives X of finite subgroups of G. Now if $K \leq G$ is a finite subgroup

$$\mathbb{Z}[K,\Omega] = \mathbb{Z}[\Omega^K] = \bigoplus_{H \in X} \mathbb{Z}[(G/H)^K]$$

But $K = gHg^{-1}$ for some $H \in X$ and $g \in G$ so $gH \in (G/H)^K$ so the augmentation map $\mathbb{Z}[-,\Omega] \longrightarrow \underline{\mathbb{Z}}$ is a surjection when evaluated at any G/K and hence is an epimorphism of $\mathcal{O}_{\mathcal{F}}$G-modules. □

Proposition 1.2.4 [KMPN09, Lemma 3.2] A right $\mathcal{O}_{\mathcal{F}}$G-module M is $\underline{\mathrm{FP}}_n$ $(n \geq 1)$ if and only if G is $\underline{\mathrm{FP}}_0$ and $M(G/K)$ is of type FP_n over the Weyl group WK for all finite subgroups $K \leq G$.

Proof. Let M be a right $\mathcal{O}_{\mathcal{F}}$G-module of type $\underline{\mathrm{FP}}_n$ and $P_* \twoheadrightarrow M$ a projective resolution, by a Bredon co-homology analogue of [Bro82, VIII4.3,4.5] we may assume that all P_i for $i \leq n$ are finitely generated free Bredon modules. Evaluating this resolution at G/H for a finite subgroup H, and applying Corollary 1.1.7, we deduce each $P_i(G/H)$ is a finite direct product of projective FP_∞ WH-modules and hence finitely generated. So we have constructed a projective resolution of $M(G/K)$ which is finitely generated up to degree n.

For the converse we use induction on n. Let $n = 0$ and M a right $\mathcal{O}_{\mathcal{F}}$G-module with $M(G/K)$ of type FP_0, ie. finitely generated, over WK. We construct a finitely generated free module F with an epimorphism $F \twoheadrightarrow M$, thus showing that M is finitely generated and hence $\underline{\mathrm{FP}}_0$.

If $H \in X$ and $K = gHg^{-1}$ then the map $K \mapsto gH$ induces a G-bijection between G/H and G/K with inverse $H \mapsto g^{-1}H$. Hence $M(G/H)$ and $M(G/K)$ are isomorphic via the maps $M(K \mapsto gH)$ and $M(H \mapsto gK)$. Similarly $\mathbb{Z}[G/K, G/H]$ and $\mathbb{Z}[G/H, G/H]$ are isomorphic via the maps $\mathbb{Z}[K \mapsto gH, G/H]$ and $\mathbb{Z}[H \mapsto g^{-1}K, G/H]$. By assumption $M(G/H)$ is finitely generated, say with a generating set of size n, choose a morphism

$$\bigoplus_1^n \mathbb{Z}[-, G/H] \longrightarrow M(-)$$

which is an epimorphism when evaluated at G/H, such a morphism can always be chosen by a Yoneda-type Lemma argument [MV03, p.9], which also tells us that we have the following commutative diagram

$$\begin{array}{ccc} \bigoplus_1^n \mathbb{Z}[G/H, G/H] & \longrightarrow & M(G/H) \\ \downarrow & & \downarrow \\ \bigoplus_1^n \mathbb{Z}[G/K, G/H] & \longrightarrow & M(G/K) \end{array}$$

where the left and right maps are bijections and the top map is an epimorphism, thus the bottom map is also an epimorphism. Hence the map $\bigoplus_1^n \mathbb{Z}[-, G/H] \longrightarrow M(-)$ is an epimorphism when evaluated at any conjugate of H. Taking a direct product of free modules of this type, one for each conjugacy class of finite subgroups, yields a finitely generated free module with an epimorphism onto $M(-)$.

Now suppose $n > 0$ and the claim is true for all $k < n$. $M(G/K)$ is a WK-module of type FP_n, so in particular it is FP_0 and finitely generated. Let $K_0 \hookrightarrow P_0 \twoheadrightarrow M$ be a short exact sequence in right $\mathcal{O}_{\mathcal{F}}$G modules with P_0 finitely generated free. By the argument of the first paragraph, for any finite subgroup H, $P_0(G/H)$ is a WH-module of type FP_∞ and by [Bie76, Proposition 1.4] $K_0(G/H)$ is FP_{n-1} and by induction, K_0 is $\underline{\mathrm{FP}}_{n-1}$. □

Corollary 1.2.5 The following are equivalent for a group G

1. G is $\underline{\mathrm{FP}}_n$.
2. G is $\underline{\mathrm{FP}}_0$ and the Weyl groups WK are FP_n for all finite subgroups K.
3. G is $\underline{\mathrm{FP}}_0$ and the centralisers $C_G K$ are FP_n for all finite subgroups K.

Proof. By the previous Proposition (1) and (2) are equivalent. To see the equivalence of (2) and (3) consider the short exact sequence

$$0 \longrightarrow K \longrightarrow N_G K \longrightarrow WK \longrightarrow 0$$

K is finite and hence FP_∞, so WK is FP_n if and only if $N_G K$ is FP_n. [Bie76, Proposition 2.7] K is finite, so $C_G K$ is finite index in $N_G K$ [Rob96, 1.6.13] and so $C_G K$ is FP_n if and only if $N_G K$ is FP_n. Combining the last two results gives WK is FP_n if and only if $C_G K$ is FP_n. □

The condition that a group G has only finitely many conjugacy classes of finite subgroups is extremely strong, in [MPN] the weaker condition quasi-$\underline{\mathrm{FP}}_n$ is introduced.

Definition 1.2.6

1. G is quasi-$\underline{\mathrm{FP}}_0$ if and only if there are finitely many conjugacy classes of finite subgroups isomorphic to a given finite subgroup.
2. G is quasi-$\underline{\mathrm{FP}}_n$ if and only if G is quasi-$\underline{\mathrm{FP}}_0$ and WK is FP_n for every finite $K \leq G$.

Let $\underline{\mathbb{Z}}_k$ be the $\mathcal{O}_\mathcal{F}$G-module defined by

$$\underline{\mathbb{Z}}_k(G/K) = \begin{cases} \mathbb{Z} & \text{if } |K| \leq k \\ 0 & \text{else.} \end{cases}$$

Proposition 1.2.7 [MPN, Lemma 6.4,6.5]

1. G is quasi-$\underline{\mathrm{FP}}_0$ if and only if $\underline{\mathbb{Z}}_k$ is finitely generated for all $k \geq 1$.
2. G is quasi-$\underline{\mathrm{FP}}_n$ if and only if $\underline{\mathbb{Z}}_k$ is $\underline{\mathrm{FP}}_n$ for all $k \geq 1$.

This proof closely mirrors that of Lemma 1.2.3.

Proof.

1. Suppose G is of type quasi-$\underline{\mathrm{FP}}_0$ and let X_k be a set of conjugacy class representatives of finite subgroups of order $\leq k$. There are only finitely many isomorphism classes groups of order $\leq k$ and the quasi-$\underline{\mathrm{FP}}_0$ property implies that for each isomorphism class of groups appearing as a subgroup of G there are only finitely many conjugacy classes of subgroups, thus X_k is finite. Set

$$\Omega_k = \coprod_{H \in X_k} G/H$$

This is a G-finite G-set and moreover $\mathbb{Z}[-, \Omega_k] \longrightarrow \underline{\mathbb{Z}}_k$ is an epimorphism by the argument at the end of Proposition 1.2.3.

For the converse, let Ω_k be a G-finite G-set with $\mathbb{Z}[-, \Omega_k] \twoheadrightarrow \Omega_k$, Ω_k necessarily has finitely many finite stabilisers and in particular the set of finite subgroups of stabilisers of Ω_k is finite. Let K be a subgroup with $|K| \leq k$ then $\mathbb{Z}[G/K, \Omega_k] \neq 0$ and by Proposition 1.1.4, K is subconjugate to one of finitely many stabilisers of Ω_k, which is a finite set as required.

□

Remark 1.2.8 In the proof of Proposition 1.2.7(1) it was actually shown that if G is quasi-$\underline{\mathrm{FP}}_0$ then we can find a free bredon module F with an epimorphism onto $\underline{\mathbb{Z}}_k$, where the stabilisers of F have order bounded by k. Moreover this "if" can be strengthened to an "if and only if" by observing that the final paragraph goes through fine if Ω_k is assumed to have stabilisers of order bounded by k.

Proof. 2. Assume that G is quasi-$\underline{\mathrm{FP}}_n$, then in particular G is quasi-$\underline{\mathrm{FP}}_0$ and by part (1) and remark 1.2.8 there is a G-finite G-set Δ_0 with stabilisers of order $\leq k$ and a short exact sequence

$$C_0(-) \hookrightarrow \mathbb{Z}[-, \Delta_0] \twoheadrightarrow \underline{\mathbb{Z}}_k$$

where $C_0(L) = 0$ for all subgroups L with $|L| > k$. By Corollary 1.1.7, $\mathbb{Z}[G/H, \Delta_0]$ is FP_∞ as a WH-module, for any finite H. When evaluated at G/H for some H with $|H| \leq k$, there is a short exact sequence

$$C_0(G/H) \hookrightarrow \mathbb{Z}[G/H, \Delta_0] \twoheadrightarrow \mathbb{Z}$$

where the central WH-module is FP_∞ and the right hand WH-module is FP_n. We deduce that $C_0(G/H)$ is FP_{n-1}. [Bie76, Proposition 1.4b]

In particular we deduce that $C_0(G/H)$ is finitely generated as a WH-module, say with a generating set of cardinality n. Then the free $\mathcal{O}_\mathcal{F}$G module $\bigoplus_1^n \mathbb{Z}[-, G/H]$ can be made to surject onto C_0 after evaluating at G/H, that this possible follows from the Yoneda-type Lemma 1.0.1 which says that any map $\mathbb{Z}[-, G/H] \to N(-)$ of $\mathcal{O}_\mathcal{F}$G modules is determined uniquely by where it sends $\mathrm{id}_{G/H} \in \mathbb{Z}[G/H, G/H]$ to in $N(G/H)$. Taking a direct product of these $\bigoplus_1^n \mathbb{Z}[-, G/H]$ for every representative H of a conjugacy class of finite subgroups with $|H| \leq k$ provides a finitely generated free $\mathcal{O}_\mathcal{F}$G-module F_0, and by the argument of Proposition 1.2.4 an epimorphism $F_0 \twoheadrightarrow C_0$.

We may now repeat the process using the short exact sequence

$$C_1(-) \hookrightarrow F_0(-) \twoheadrightarrow C_0(-)$$

$F_0(G/H)$ is again FP_∞ and $C_0(G/H)$ is FP_{n-1} so $C_1(G/H)$ is FP_{n-2}, again by [Bie76, Proposition 1.4b]. It is clear that by induction this will yield a partial projective resolution of length n with each term finitely generated.

For the converse let K be a finite subgroup of G with $|K| = k$. Then $\mathbb{Z}_k(G/K) \cong \mathbb{Z}$ is FP_n as a WK-module by the argument of the first paragraph of Proposition 1.2.4.

□

Let $\mathcal{F}_k$ be the sub-family of subgroups of $\mathcal{F}$ whose orders are bounded by k.

Corollary 1.2.9 [MPN, Lemma 6.7] A group is $\underline{\mathrm{FP}}_n$ if and only if it is FP_n over $\mathcal{O}_{\mathcal{F}_k}G$ for every k.

Proof. This is purely a restatement of the results of Proposition 1.2.7 and remark 1.2.8. □

Remark 1.2.10 In [MPN] a concept of quasi-$\underline{F}_\infty$ is also defined and related to quasi-$\underline{\mathrm{FP}}_\infty$.

2 Centralisers in Houghton's Groups

Houghton's group H_n was first defined by Houghton while investigating the first cohomology of groups with permutation module coefficients [Hou78]. He gave it as an example of a group H_n acting on a set S with $H^1(H_n, A \otimes \mathbb{Z}[S]) = A^{n-1}$ for any Abelian group A.

In [Bro87], Brown used an important new technique to show that the Thompson-Higman groups $F_{n,r}, T_{n,r}$ and $V_{n,r}$ were FP_∞. In the same paper he shows that Houghton's group H_n is interesting from the viewpoint of cohomological finiteness conditions, namely H_n is FP_{n-1} but not FP_n. Thompsons group F was previously shown by different methods to be FP_∞ [BG84], thus providing the first known example of a torsion-free FP_∞ group with infinite co-homological dimension.

There has been recent interest in the structure of the centralisers of Thompson's groups, in [MPN] the centralisers of finite subgroups of generalisations of Thompsons groups T and V are calculated and this data is used to give information about Bredon (co-)homological finiteness conditions satisfied by these groups. The results obtained in [MPN, Theorem 4.4, 4.8] have some similarity with those obtained here. In [BBG$^+$], a description of centralisers of elements in the Thompson-Higman group V_n is given.

This Section is organised as follows: Section 2.1 contains an analysis of the centralisers of finite subgroups in Houghton's group. As Corollary 2.1.4 we obtain that centralisers of finite subgroups are FP_{n-1} but not FP_n. This should be compared with [KMPN11] where examples are given of soluble groups of type FP_n with centralisers of finite subgroups that are not FP_n, and also with [KMPN10], where it is shown that centralisers of finite subgroups in soluble groups of type $\underline{\mathrm{FP}}_\infty$ (often denoted Bredon-FP_∞) are always of type FP_∞.

In Section 2.2 our analysis is extended to arbitrary elements and virtually cyclic subgroups. Using this information elements in H_n are constructed whose centralisers are FP_i for any $0 \leq i \leq n-3$. In section 2.3, the construction of Brown [Bro87] used to prove that H_n is FP_{n-1} but not FP_n is shown to be a model for $\underline{\mathrm{E}}\, H_n$, the classifying space for proper actions of H_n. Finally Section 2.4 contains a discussion of Bredon (co-)homological finiteness conditions are satisfied by Houghton's group. Namely we show in Proposition 2.4.1 that H_n is not quasi-$\underline{\mathrm{FP}}_0$ and in Proposition 2.4.3 that the Bredon cohomological dimension and the Bredon geometric dimension with respect to the family of finite subgroups are both equal to n.

Fixing a natural number $n > 1$, define *Houghton's group* H_n to be the group of permuatations of $S = \mathbb{N} \times \{1, \ldots, n\}$ which are "eventually translations", ie. for any given permuation $h \in H_n$ there are collections $\{z_1, \ldots, z_n\} \in \mathbb{N}^n$ and $\{m_1, \ldots, m_n\} \in \mathbb{Z}^n$ with

$$h(i, x) = (i + m_x, x) \text{ for all } x \in \{1, \ldots, n\} \text{ and all } i \geq z_x \tag{2}$$

Define a map ϕ as follows:

$$\phi : H \to \{(m_1, \ldots, m_n) \in \mathbb{Z}^n \,:\, \sum m_i = 0\} \cong \mathbb{Z}^{n-1} \tag{3}$$

$$\phi : h \mapsto (m_1, \ldots, m_n) \tag{4}$$

It's kernel is exactly the permutations which are "eventually zero" on S, ie. the infinite symmetric group Sym_∞ (the finite support permutations of a countable set).

From now on we fix an n and write H instead of H_n, unless the value of n needs to be explicitly mentioned.

2.1 Centralisers of Finite Subgroups in H

First we recall some properties of group actions on sets, before specialising to Houghton's group.

Proposition 2.1.1 If G is a group acting on a countable set X and H is any subgroup of G then

1. If x and y are in the same G-orbit then their isotropy subgroups G_x and G_y are G-conjugate.
2. If $g \in C_G(H)$ then $H_{gx} = H_x$ for all $x \in X$.
3. Partition X into $\{X_i\}_{i=1}^n$, where $n \in \mathbb{N} \cup \{\infty\}$, via the equivalence relation $x \sim y$ if and only if H_x is H-conjugate to H_y. Any two points in the same H-orbit will lie in the same partition and any $c \in C_G(H)$ maps X_i onto X_i for all i.
4. Let G act faithfully on X, with the property that for all $g \in G$ and $X_i \subseteq X$ as in the previous section, there exists a group element $g_i \in G$ which fixes $X \setminus X_i$ and acts as g does on X_i. Then $C_G(H) = C_1 \times \cdots \times C_n$ where C_i is the subgroup of $C_G(H)$ acting trivially on $X \setminus X_i$.

Proof. (1) and (2) are standard results.

3. This follows immediately from (1) and (2).
4. This follows from (3) and our new assumption on G: Let $c \in C_G(H)$ and c_i be the element given by the assumption. Since the action of G on X is faithful, c_i is necessarily unique. That the action is faithful also implies $c = c_1 \cdots c_n$ and that any two c_i and c_j commute in G because they act non-trivially only on distinct X_i. Thus we have the necessary isomorphism $C_G(H) \longrightarrow C_1 \times \cdots \times C_n$.

□

Let $Q \leq H$ be a finite subgroup of Houghton's group H and $S_Q = S \setminus S^Q$ the set of points of S which are *not fixed* by Q. Q being finite implies $\phi(Q) = 0$ as any element q with $\phi(q) \neq 0$ necessarily has infinite order. For every $q \in Q$ there exists $\{z_1, \ldots, z_n\} \in \mathbb{N}^n$ such that

$$q(i,x) = (i,x) \text{ if } i \geq z_x$$

Taking z_i' to be the maximum of these z_i over all elements in Q, then Q must fix the set $\{(i,x) \,:\, i \geq z_x'\}$ and in particular $S_Q \subseteq \{(i,x) \,:\, i < z_x'\}$ is finite.

We need to see that the subgroup $Q \leq H$ acting on the set S satisfies the conditions of Proposition 2.1.1(4). We give the following Lemma in more generality than is needed here, as it will come in useful later on. That the action is faithful is automatic as an element $h \in H$ is uniquely determined by its action on the set S.

Lemma 2.1.2 Let $Q \leq H$ be a subgroup, which is either finite or the form $F \rtimes \mathbb{Z}$ for F a finite subgroup of H. Partition S with respect to Q into sets $\{S_i\}$ as in Proposition 2.1.1(3). Then the conditions of Proposition 2.1.1(4) are satisfied.

Proof. Fix $j \in \{1, \ldots, n\}$ and let h_j denote the permutation of S which fixes $S \setminus S_j$ and acts as h does on S_j. We wish to show that h_j is an element of H.

There are only finitely many elements in Q with finite order so as in the argument just before this Lemma we may choose integers z_i for $i \in \{1, \ldots, n\}$ such that if q is a finite order element of Q then $q(i,x) = (i,x)$ whenever $i \geq z_i$. If Q is a finite group then either:

- S_j is fixed by Q, in which case

$$\{(i,x) \, : \, i \geq z_x \, , \, x \in \{1,\ldots,n\}\} \subseteq S_j$$

so $h_j(i,x) = h(i,x)$ for all $i \geq z_x$. In particular for large enough i, h_j acts as a translation on (x,i) and is hence an element of H.

Or

- S_j is not fixed by Q, in which case

$$S_j \subseteq \{(i,x) \, : \, i < z_x \, , \, x \in \{1,\ldots,n\}\}$$

In particular S_j is finite and $h_j(i,x) = (i,x)$ for all $i \geq z_x$. Hence h_j is an element of H.

It remains to treat the case where $Q = F \rtimes \mathbb{Z}$. Write w for a generator of $\mathbb{Z}$ in $F \rtimes \mathbb{Z}$. By choosing a larger z_i if needed we may assume w acts either trivially or as a translation on (i,x) whenever $i \geq z_x$. Hence for any $x \in \{1,\ldots,n\}$, the isotropy group in Q of $\{(i,x) \, : \, i \geq z_x\}$ is either F or Q.

If S_j has isotropy group Q or F then for some $x \in \{1,\ldots,n\}$, either:

- $$S_j \cap \{(i,x) : i \geq z_x\} = \{(i,x) : i \geq z_x\}$$

  In which case $h_j(i,x) = h(i,x)$ for $i \geq z_x$. In particular for large enough i, h_j acts as a translation on (i,x) and hence is an element of H.

Or

- $$S_j \cap \{(i,x) : i \geq z_x\} = \emptyset$$

  In which case $h_j(i,x) = (i,x)$ for $i \geq z_x$. In particular for large enough i, h_j fixes (i,x) and hence is an element of H.

If S_j is the set corresponding to an isotropy group not equal to F or Q then

$$S_j \subseteq \{(i,x) \, : \, i \geq z_x \, , \, x \in \{1,\ldots,n\}\}$$

So h_j fixes (i,x) for $i \geq z_x$ and hence h_j is an element of H. □

Partition S into disjoint sets as in Proposition 2.1.1(3), the set with isotropy in Q equal to Q is S^Q and since S_Q is finite the partition is finite, thus

$$S = S^Q \cup S_1 \cup \cdots \cup S_t$$

Proposition 2.1.1(4) gives that

$$C_H(Q) = H|_{S^Q} \times C_1 \times \ldots \times C_t$$

where each C_i acts only on S_i and leaves S^Q and S_j fixed for $i \neq j$. The first element of the direct product decomposition is the subgroup of $C_H(Q)$ acting only on S^Q and leaving $S \setminus S^Q$ fixed. This is $H|_{S^Q}$ (H restricted to S^Q) because, as the action of Q on S^Q is trivial, any permutation of S^Q will centralise Q. Choose a bijection $S^Q \to S = \mathbb{N} \times \{1,\ldots,n\}$ such that for all i, $(x,i) \mapsto (x + a_i, i)$ for large enough x and some $a_i \in \mathbb{Z}$, this induces an isomorphism between $H|_{S^Q}$ and H.

The C_i are finite as they are subgroups of the permutation group of the finite set S_i, we give an explicit description of them as extensions of a finite group by a symmetric group. S_i is finite and hence splits as a union of finitely many Q-orbits. Choose representatives $\{s_1,\ldots,s_r\} \subset S_i$ for these orbits. These s_i can be chosen to have the same Q-stabilisers: If $Q_{s_1} \neq Q_{s_2}$ there is some $q \in Q$ such that $Q_{qs_2} = qQ_{s_2}q^{-1} = Q_{s_1}$

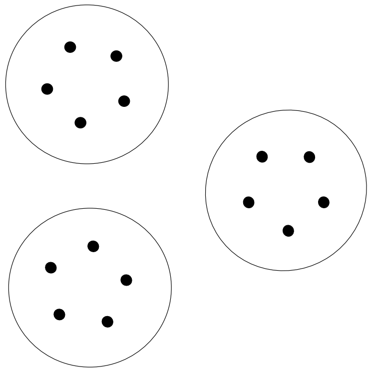

Figure 1: A representation of S_i. The large circles are the sets $\{Qs_1 \ldots, Qs_r\}$ (in this figure $r = 3$). Elements $\alpha(\sigma) \in \mathrm{Sym}_r$ permute only the large circles, while elements of $\mathrm{Ker}\,\pi \leq \mathrm{Sym}_m^r$ leave the large circles fixed and permute only elements inside them.

(the partitions S_i were chosen to have this property by Proposition 2.1.1). s_2 can then be replaced by qs_2 and by iterating this process we get a set of representatives with the required property.

Define a map

$$\begin{aligned} \alpha : \mathrm{Sym}_r &\to C_i \\ \sigma &\mapsto \big(\alpha(\sigma) : qs_i \mapsto qs_{\sigma(i)} \text{ for all } q \in Q\big) \end{aligned}$$

Each $\alpha(\sigma)$ is a well defined element of H since

$$qs_i = \tilde{q}s_i \Leftrightarrow \tilde{q}^{-1}q \in Q_{s_i} = Q_{s_{\sigma(j)}} \Leftrightarrow qs_{\sigma(i)} = \tilde{q}s_{\sigma(i)}$$

Note that this implies $\alpha(\sigma)q = q\alpha(\sigma)$ for all $q \in Q$, ie. $\alpha(\sigma) \in C_H(Q)$.

If $x \in C_i$ then x permutes the Q-orbits $\{Qs_1, \ldots, Qs_r\}$ of S_i, inducing an action of the symmetric group Sym_r on $\{Qs_1, \ldots, Qs_r\}$, this defines a map $\pi : C_i \to \mathrm{Sym}_r$. By definition, $\alpha(\sigma)$ takes Qs_i onto $Qs_{\sigma(i)}$ and so $\pi \circ \alpha(\sigma) = \sigma$. Since π is split by α, there is a split short exact sequence

$$0 \longrightarrow K \longrightarrow C_i \underset{\pi}{\overset{\alpha}{\rightleftarrows}} \mathrm{Sym}_r \longrightarrow 0$$

where the kernel $K = \mathrm{Ker}\,\pi$ takes each Q-orbit Qs_i to itself, but may permute the elements inside those Q-orbits. If $k \in K$ then defining $k_i \in K$ to act on Qs_i as k does and to fix Qs_j for all $i \neq j$ yields $k = k_1 \cdots k_r$ where k_i and k_j commute for all i, j since they act non-trivially only on disjoint sets. As each element k_i acts non-trivially only on Qs_i, such elements can be regarded as a subgroup of Sym_m where $m = |Q/Q_{s_i}|$. Hence $\mathrm{Ker}\,\pi \leq \mathrm{Sym}_m^r$ (direct product of r copies of Sym_m) . To summarise:

Proposition 2.1.3 The centralisers $C_H(Q)$ of any finite subgroup $Q \leq H$ split as a product

$$C_H(Q) = H|_{S^Q} \times C_1 \times \cdots \times C_t$$

where $H|_{S^Q} \cong H$ is Houghton's group restricted to S^Q and each C_i fits into a split short exact sequence

$$0 \longrightarrow K \longrightarrow C_i \longrightarrow \mathrm{Sym}_r \longrightarrow 0$$

where $K \leq \mathrm{Sym}_m^r$, $m = |Q/Q_i|$ (Q_i an isotropy group of S_i) and $r = |S_i|/m$. In particular H is finite index in $C_H(Q)$.

Corollary 2.1.4 If Q is a finite subgroup of H then the centraliser $C_H(Q)$ is FP_{n-1} but not FP_n.

Proof. H is finite index in the centraliser $C_H(Q)$ by Proposition 2.1.3. Appealing to Brown's result [Bro87, 5.1] that H is FP_{n-1} but not FP_n, and that a group is FP_n if and only if a finite index subgroup is FP_n [Bro82, VIII.5.5.1] we can deduce $C_H(Q)$ is FP_{n-1} but not FP_n. □

2.2 Centralisers of Elements in H

If $q \in H$ is an element of finite order then the subgroup $Q = \langle q \rangle$ is a finite subgroup and the previous section may be used to describe the centraliser $C_H(q) = C_H(Q)$. Thus for an element q of finite order $C_H(q) \cong C \times H$ for some finite group C.

If $q \in H$ is an element of infinite order and $Q = \langle q \rangle$ then we may apply Proposition 2.1.1(3) to split up S into a disjoint collection $\{S_i \ : \ i \in I \subseteq \mathbb{N}\} \cup S^Q$ (S^Q is the element of the collection associated to the isotropy group Q). Assume that S_0 is the set associated to the trivial isotropy group. Since q is a translation on $(i, x) \in S = \{1, \dots, n\} \times \mathbb{N}$ for large enough x and points acted on by such a translation have trivial isotropy, there are only finitely many elements of S whose isotropy group is neither the trivial group nor Q. Hence S_i is finite for $i \neq 0$ and the set I is finite. From now on let $I = \{0, \dots, t\}$. We now use Lemma 2.1.2 and Proposition 2.1.1(4) as in the previous section: $C_H(Q)$ splits as

$$C_H(Q) \cong C_0 \times C_1 \times \cdots \times C_t \times H|_{S^Q}$$

Where C_i acts only on S_i and $H|_{S^Q}$ is Houghton's group restricted to S^Q. Unlike in the last section, $H|_{S^Q}$ may not be isomorphic to H. Let $J \subseteq \{1, \dots, n\}$ satisfy

$$i \in J \text{ if and only if } (x, i) \in S^Q \text{ for all } x \geq z_i, \text{ some } z_i \in \mathbb{N}$$

If $i \notin J$ then for large enough x, q must act as a non-trivial translation on (x, i), and the set $(\{i\} \times \mathbb{N}) \cap S^Q$ is finite. Clearly $|J| \leq n - 2$, but different elements q may give values $0 \leq |J| \leq n - 2$. In the case $|J| = 0$, S^Q is necessarily finite and so $H|_{S^Q}$ is isomorphic to a finite symmetric group on S^Q. It is also possible that $S^Q = \emptyset$, in which case $H|_{S^Q}$ is just the trivial group. If $|J| \neq 0$ then the argument proceeds now as in the previous section by choosing a bijection

$$S^Q \to J \times \mathbb{N}$$

such that $(x, i) \mapsto (x + a_i, i)$ for some $a_i \in \mathbb{Z}$ whenever x is large enough and $i \in J$. This set map induces a group isomorphism between $H|_{S^Q}$ and $H_{|J|}$ (Houghton's group on the set $J \times \mathbb{N}$).

The arguments of Proposition 2.1.3 show that C_i is finite for $i \neq 0$ so it remains to treat C_0, the subgroup of the centraliser acting only on S_0. Recall that S_0 is the element of the partition with trivial isotropy group.

Let $(m_1, \dots, m_n) \in \mathbb{Z}^n, (z_1, \dots, z_n) \in \mathbb{N}^n$ be such that

$$q(i, x) = (i + m_x, x) \text{ if } i \geq z_x$$

Consider orbit representatives for the Q orbits in S_0, there can be no more than $z_i + m_i$ orbit representatives on $(\{i\} \times \mathbb{N}) \cap S_0$ and hence only finitely many Q-orbits of S_0. Choose representatives $\{s_1, \dots, s_r\} \subset S_0$ for these Q-orbits and, as in Section 2.1, define a map $\alpha : \mathrm{Sym}_r \to C_0$ by

$$\alpha(\sigma) : q^i s_i \mapsto q^i s_{\sigma(i)} \text{ for all } i$$

as before this implies that $\alpha(\sigma) \in C_H(q)$, and α is split by the map $\pi : C_0 \to \mathrm{Sym}_r$, induced by the action of C_0 on the set $\{Qs_1, \ldots, Qs_r\}$.

The kernel of π consists of elements centralising Q, fixing $S \setminus S_0$, and taking Q-orbits to themselves in S_0. If $h \in \mathrm{Ker}\,\pi$ and $hs_i = q^{j_i} s_i$ for some $j_i \in \mathbb{Z}$ then

$$hq^m s_i = q^m hs_i = q^{j_i+m} s_i$$

Thus h can be defined purely be a collection of integers $j_1, \ldots, j_r$, where h acts as

$$hq^m s_i = q^{m+j_i} s_i \quad \text{for all } 1 \leq i \leq r \text{ and } m \in \mathbb{Z}$$

Writing K for $\mathrm{Ker}\,\pi$, we deduce $K \leq \mathbb{Z}^r$ and that there is a split short exact sequence

$$0 \longrightarrow K \longrightarrow C_0 \longrightarrow \mathrm{Sym}_r \longrightarrow 0$$

To summarise:

Theorem 2.2.1 1. If $q \in H$ is an element of finite order then

$$C_H(q) \cong C \times H$$

for a finite group C and a copy of Houghton's group $H = H_n$.

2. If $q \in H$ is an element of infinite order then either

$$C_H(q) \cong C_0 \times C \times H_i$$

or

$$C_H(q) \cong C_0 \times C$$

where C is a finite group, H_i is Houghton's group with $1 \leq i \leq n-2$ and $C_0 = K \rtimes \mathrm{Sym}_r$ for some $K \leq \mathbb{Z}^r$ and natural number r.

In Corollary 2.1.4 it was proved that for an element q of finite order, $C_H(q)$ is FP_{n-1} but not FP_n. The situation is much worse for elements q of infinite order, in which case the centraliser may not even be finitely generated, for example when n is odd and q is the element acting on $S = \mathbb{N} \times \{1, \ldots, n\}$ as

$$q : \begin{cases} (i, x) \mapsto (i+1, x) & \text{if } x \leq (n-1)/2 \\ (i, x) \mapsto (i-1, x) & \text{if } (n+1)/2 \leq x \leq n-1 \text{ and } i \neq 0 \\ (0, x) \mapsto (0, x - ((n-1)/2)) & \text{if } (n+1)/2 \leq x \leq n-1 \\ (i, n) \mapsto (i, n) & \end{cases}$$

then the only fixed points are on the ray $\mathbb{N} \times \{n\}$. The argument leading up to Theorem 2.2.1 shows that the centraliser is a direct product of groups, one of which is Houghtons group H_1 which is isomorphic to the infinite symmetric group and hence not finitely generated. In particular for this q, the centraliser $C_H(q)$ is not even FP_1. A similar example can easily be constructed when n is even.

All the groups in the direct product decomposition from Theorem 2.2.1 except H_i are FP_∞, being built by extensions from finite groups and free abelian groups. By choosing various infinite order elements q, for example by modifying the example of the previous paragraph, the centralisers can be chosen to be FP_i for $0 \leq i \leq n-3$. The upper bound of $n-3$ arises because any infinite order element q must necessarily be "eventually a translation" (in the sense of (2)) on $\{i\} \times \mathbb{N}$ for *at least* two i. As such the copy of Houghton's group H in the centraliser can act on at most $n-2$ rays and is thus at largest H_{n-2}, which is FP_{n-3}.

Corollary 2.2.2 If Q is an infinite virtually cyclic subgroup of H then

$$C_H(Q) \cong C_0 \times C \times H_i$$

or

$$C_H(Q) \cong C_0 \times C$$

where C is a finite group, H_i is Houghton's group with $1 \leq i \leq n-2$ and $C_0 = K \rtimes \mathrm{Sym}_r$ for some $K \leq \mathbb{Z}^r$ and natural number r.

This Corollary can be proved by reducing to the case of Theorem 2.2.1, but before that we require the following Lemma.

Lemma 2.2.3 Every infinite virtually cyclic subgroup Q of H is finite-by-$\mathbb{Z}$.

Proof. By [JPL06, Proposition 4], Q is either finite-by-$\mathbb{Z}$ or finite-by-D_∞ where D_∞ denotes the infinite dihedral group, we show the latter cannot occur. Assume that there is a short exact sequence

$$0 \longrightarrow F \hookrightarrow Q \xrightarrow{\pi} \mathrm{D}_\infty \longrightarrow 0$$

regarding F as a subgroup of Q. Let a, b generate D_∞, so that

$$\mathrm{D}_\infty = \langle a, b \mid a^2 = b^2 = 1 \rangle$$

Let $p, q \in Q$ be lifts of a, b, such that $\pi(p) = a$, $\pi(q) = b$, then $p^2 \in F$. Since F is finite, p^2 has finite order and hence p has finite order. The same argument shows that q has finite order. $pq \in Q$ necessarily has infinite order as $\pi(pq)$ is infinite order in D_∞.

However, since p and q are finite order elements of H, by the argument at the beginning of Section 2.1 they both permute only a finite subset of S. Thus pq permutes a finite subset of S and is of finite order, but this contradicts the previous paragraph.

□

Proof of Corollary 2.2.2. Using the previous Lemma, write Q as $Q = F \rtimes \mathbb{Z}$ where F is a finite group. As F is finite, the set S_F of points not fixed by F is finite (see the argument at the beginning of Section 2.1). Let $z \in \mathbb{N}$ be such that for $i \geq z$, F acts trivially on (i, x) for all x, and $\mathbb{Z}$ acts on (i, x) either trivially or as a translation. Applying Lemma 2.1.2 and Proposition 2.1.1, S splits as a disjoint union

$$S = S^Q \cup S_0 \cup S_1 \cup \cdots \cup S_t$$

where S^Q is the fixed point set, S_0 is the set with isotropy group F and the S_i for $1 \leq i \leq t$ are subsets of $\{(i, x)\ i \leq z\}$, and hence all finite. By Proposition 2.1.1, $C_H(Q)$ splits as a direct product

$$C = H|_{S^Q} \times C_0 \times C_1 \times \ldots \times C_t$$

where $H|_{S^Q}$ denotes Houghton's group restricted to S_Q. The argument of Theorem 2.2.1 showing that $H|_{S^Q} \cong H_i$ for some $1 \leq i \leq n-2$ goes through with no change, as does the proof that the groups C_i are all finite for $1 \leq i \leq t$. It remains to observe that because every element in S_0 is fixed by F, any element of H centralising $\mathbb{Z}$ and fixing $S \setminus S_0$ necessarily also centralises Q and is thus a member of C_0. This reduces us again to the case of Theorem 2.2.1 showing that $C_0 \cong K \rtimes \mathrm{Sym}_r$, where $K \leq \mathbb{Z}^r$ for some natural number r. □

2.3 Brown's Model for $\underline{E}H$

The main result of this section will be Corollary 2.3.4, where the construction of Brown [Bro87] used to prove that H is FP_{n-1} but not FP_n is shown to be a model for $\underline{E}\,H$.

Since the main objects of study in this section are monoids, maps are written from left to right.

Write $\mathcal{M}$ for the monoid of injective maps $S \to S$ with the property that every permutation is "eventually a translation" (in the sense of (2)), and write T for the free monoid generated by $\{t_1, \dots, t_n\}$ where

$$(i,x)t_j = \begin{cases} (i+1,x) & \text{if } i = j \\ (i,x) & \text{if } i \neq j \end{cases}$$

The elements of T will be called *translations*. The map $\phi : H \to \mathbb{Z}^n$, defined in (3), extends naturally to a map $\phi : \mathcal{M} \to \mathbb{Z}^n$. Give $\mathcal{M}$ a poset structure by setting $\alpha \leq \beta$ if $\beta = t\alpha$ for some $t \in T$. The monoid $\mathcal{M}$ can be given the obvious action on the right by H, which in turn gives an action of H on the poset $(\mathcal{M}, \leq)$ since $\beta = t\alpha$ implies $\beta h = t\alpha h$ for all $h \in H$. Let $|\mathcal{M}|$ be the geometric realisation of this poset, namely simplicies in $|\mathcal{M}|$ are ordered collections of elements in $\mathcal{M}$ with the obvious face maps. An element $h \in H$ fixes a vertex $\{\alpha\} \in |\mathcal{M}|$ if and only if $s\alpha h = s\alpha$ for all $s \in S$ if and only if h fixes $S\alpha$, so the stabiliser H_α may only permute the finite set $S \setminus S\alpha$ and we may deduce

Proposition 2.3.1 Stabilisers of simplicies in $|\mathcal{M}|$ are finite.

We now build up to the the proof that $|\mathcal{M}|$ is a model for $\underline{E}\,H$ with a few Lemmas.

Proposition 2.3.2 If $Q \leq H$ is a finite group then the fixed point set $|\mathcal{M}|^Q$ is non-empty and contractible.

Proof. For all $q \in Q$, choose $\{z_0(q), \dots, z_n(q)\}$ to be an n-tuple of natural numbers such that $(i,x)q = (i,x)$ whenever $i \geq z_x(q)$ for all i. Q then fixes all elements $(i,x) \in S$ with $i \geq \max_Q z_x(q)$. Define a translation $t = t_1^{\max_Q z_1(q)} \cdots t_n^{\max_Q z_n(q)}$, $t \in \mathcal{M}^Q$ so $\{t\}$ is a vertex of $|\mathcal{M}|^Q$ and $|\mathcal{M}|^Q \neq \emptyset$.

If $\{m\}, \{n\} \in |\mathcal{M}|^Q$ then let $a, b \in T$ be two translations such that

$$\phi(m) - \phi(n) = \phi(b) - \phi(a)$$

(recall that for a translation t, $\phi(t)$ must be an n-tuple of positive numbers). Thus $\phi(am) = \phi(bn)$, and since $am, bn \in \mathcal{M}$ there exist n-tuples $\{z_1, \dots, z_n\}$ and $\{z'_1, \dots, z'_n\}$ such that am acts as a translation for all $(i,x) \in S$ with $i \geq z_x$ and bn acts as a translation for all $(i,x) \in S$ with $i \geq z'_x$. Let

$$c = t_1^{\max\{z_1, z'_1\}} \dots t_n^{\max\{z_n, z'_n\}}$$

so that $cam = cbn$, further pre-composing c with a large translation (for example that from the first section of this proof) we can assume that $cam = cbn \in \mathcal{M}^Q$, and $\{cam = cbn\} \in |\mathcal{M}|^Q$. This shows there is a cone over any two elements in $|\mathcal{M}|^Q$ and hence $|\mathcal{M}|^Q$ is contractible. □

Proposition 2.3.3 If $Q \leq H$ is an infinite group then $|\mathcal{M}|^Q = \emptyset$.

Proof. Consider an infinite subgroup $Q \leq H$ with $|\mathcal{M}|^Q \neq \emptyset$ and choose some vertex $\{m\} \in |\mathcal{M}|^Q$. For any $q \in Q$, since $mq = m$ it must be that $\phi(m) + \phi(q) = \phi(m)$ and $\phi(q) = 0$, hence Q is a subgroup of $\mathrm{Sym}_\infty \leq H$. Furthermore Q must permute an infinite subset of S (if it permuted just a finite set it would be a finite subgroup). $mq = m$ implies that this infinite subset is a subset of $S \setminus Sm$ but this is finite by construction. So the fixed point subset $|\mathcal{M}|^Q$ for any infinite subgroup Q is empty. □

Corollary 2.3.4 $|\mathcal{M}|$ is a model for $\underline{\mathrm{E}}\,H$.

Proof. Combine Propositions 2.3.1, 2.3.2 and 2.3.3. □

2.4 Finiteness Conditions satisfied by H

Recall from Section 1.2 that a group G is $\underline{\mathrm{FP}}_0$ if and only if it has finitely many conjugacy classes of finite subgroups. G satisfies the weaker quasi-$\underline{\mathrm{FP}}_0$ condition if and only if it has finitely many conjugacy classes of subgroups isomorphic to a given finite subgroup.

Proposition 2.4.1 H is not quasi-$\underline{\mathrm{FP}}_0$.

Before the above Proposition is proved, we need a lemma. In the infinite symmetric group Sym_∞ acting on the set S, elements can be represented by products of disjoint cycles. We use the standard notation for a cycle: $(s_1, s_2, \ldots, s_m)$ representes the element of Sym_∞ sending $s_i \mapsto s_{i+1}$ for $i < n$ and $s_n \mapsto s_1$. Any element of finite order in H is contained in the infinite symmetric group Sym_∞ by the argument at the beginning of Section 2.1. We say two elements of Sym_∞ have the same *cycle type* if they have the same number of cycles of length m for each $m \in \mathbb{N}$.

Lemma 2.4.2 If q is a finite order element of H and h is an arbitrary element of H, then hqh^{-1} is the permutation given in the disjoint cycle notation by applying h to each element in each disjoint cycle of q. In particular, if q is represented by the single cycle $(s_1, \ldots s_m)$, then hqh^{-1} is represented by $(hs_1, \ldots, hs_m)$.

Furthermore, two finite order elements of H are conjugate if and only if they have the same cycle type.

Proof. The proof of the first part is analagous to [Rot95, Lemma 3.4]. Let q be an element of finite order and h an arbitrary element of H. If q fixes $s \in S$ then hqh^{-1} fixes hs. If $q(i) = j$, $h(i) = k$ and $h(j) = l$, for $i, j, k, l \in S$, then $hqh^{-1}(k) = l$ exactly as required.

By the above, conjugate elements have the same cycle type. For the converse, notice any two finite order elements with the same cycle type necessarily lie in Sym_r for some $r \in \mathbb{N}$ so by [Rot95, Theorem 3.5] they are conjugated by an element of Sym_r. □

Proof of Proposition 2.4.1. If q is any order 2 element of H, then $\{\mathrm{id}_H, q_1\}$ is a subgroup of H isomorphic to $\mathbb{Z}_2$. Choosing a collection of elements q_i for each $i \in \mathbb{N}_{\geq 1}$, so that q_i has i disjoint 2-cycles gives a collection of isomorphic subgroups which are all non-conjugate by Lemma 2.4.2. □

Proposition 2.4.3 $\underline{\mathrm{cd}}\, H = \underline{\mathrm{gd}}\, H = n$

Proof. The argument below is a variation of [Gan11, Proposition 3.40]. As described in the introduction, H can be written as

$$\mathrm{Sym}_\infty \hookrightarrow H \twoheadrightarrow \mathbb{Z}^{n-1}$$

$\underline{\mathrm{gd}}\, \mathbb{Z}^{n-1} = n - 1$ since a model for $\underline{\mathrm{E}}\, \mathbb{Z}^{n-1}$ is given by $\mathbb{R}^{n-1}$ with the obvious action. $\underline{\mathrm{gd}}\, \mathrm{Sym}_\infty = 1$ by [LW12, Theorem 4.3], as it is the co-limit of its finite subgroups each of which have geometric dimension 0, and the directed category over which the co-limit is taken has homotopy dimenion 1 [LW12, Lemma 4.2]. $\mathbb{Z}^{n-1}$ is torsion free and so has a bound of 1 on the orders of its finite subgroups and we deduce from [Lü00, Theorem 3.1] that $\underline{\mathrm{gd}}\, H \leq n - 1 + 1 = n$.

To deduce the other bound, assume that $\underline{\mathrm{cd}}\, G \leq n - 1$. By [BLN01, Theorem 2]

$$\mathrm{cd}_{\mathbb{Q}} \leq \underline{\mathrm{cd}}\, G = n - 1$$

In [Bro87, Theorem 5.1], it is proved that H is FP_{n-1} (but not FP_n), combining this with [LN01a, Proposition 1] we deduce that there is a bound on the orders of the finite subgroups of H, but this is obviously a contradiction. Thus

$$n \leq \underline{\mathrm{cd}}\, H \leq \underline{\mathrm{gd}}\, H \leq n$$

□

Remark 2.4.4 In [DP12, Theorem 1], it is proved that for every elementary amenable group G of finite hirsch length h and cardinality $\aleph_0$, $\underline{\underline{\mathrm{gd}}}\, G \leq n+h+2$, (see the beginning of [HL92] for a precise definition of Hirsch length for elementary amenable groups). From this we may deduce that since the hirsch length of H is $h(H) = n - 1$,

$$\underline{\underline{\mathrm{gd}}}\, H \leq n + 1$$

In [LW12, Corollary 5.4], it is proved that $\underline{\underline{\mathrm{gd}}}\, G \geq \underline{\mathrm{gd}}\, G - 1$ for any group G. Thus we deduce

$$n - 1 \leq \underline{\underline{\mathrm{gd}}}\, H \leq n + 1$$

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