Research Statement

Simon St. John-Green

Position: PhD Student, University of Southampton.
Email: Simon.StJG@gmail.com
Web: www.personal.soton.ac.uk/ssg1g11

My research interests are focused around proper actions on spaces and associated invariants, often combining methods from geometric group theory, homological algebra and group cohomology.

1. Cohomological Invariants and Proper Actions

If G is a discrete group and $\mathfrak{F}$ a family of subgroups of G, then a G-CW complex X is a model $\mathrm{E}_{\mathfrak{F}}\mathrm{G}$ if X^H is contractible for all $H \in \mathfrak{F}$ and X^H is empty for all other subgroups of G, such spaces are unique up to G-homotopy equivalence. There has recently been a lot of interest in there spaces because they appear in the Baum–Connes and Farrell–Jones conjectures and an important invariant is the geometric dimension, denoted $\mathrm{gd}_{\mathfrak{F}} G$ — the minimal dimension of a model for $\mathrm{E}_{\mathfrak{F}}\mathrm{G}$.

Let n_G denote the minimal dimension of a proper contractible G-CW complex. Kropholler and Mislin have conjectured that if $n_G < \infty$ then $\mathrm{gd}_{\mathfrak{F}} G < \infty$, where $\mathfrak{F}$ is the family of finite subgroups of G, however since n_G and $\mathrm{gd}_{\mathfrak{F}}$ are quite intractible it is useful to study related algebraic invariants. The correct algebraic mirror of the geometric dimension is the Bredon cohomological dimension $\mathrm{cd}_{\mathfrak{F}} G$, which almost always coincides with $\mathrm{gd}_{\mathfrak{F}} G$ [12][10], the only exception being some groups with $\mathrm{cd}_{\mathfrak{F}} G = 2$ but $\mathrm{gd}_{\mathfrak{F}} G = 3$[3].

There have been many invariants put forward to mirror n_G. There is the the $\mathfrak{F}$-cohomological dimension, $\mathfrak{F}\mathrm{cd}\, G$, suggested by Nucinkis [15], the Bredon cohomology with coefficients restricted to the family of Mackey functors, $\mathcal{M}_{\mathfrak{F}}\mathrm{cd}\, G$ [14] or the Bredon cohomology with coefficients restricted to the family of cohomological Mackey functors, $\mathcal{H}_{\mathfrak{F}}\mathrm{cd}\, G$ [9], and finally Bahlekeh, Dembegioti and Talelli have suggested the Gorenstein cohomological dimension, $\mathrm{Gcd}\, G$ [1].

These algebraic invariants fit together as follows:

$$(\star) \qquad \mathrm{Gcd}\, G \leq \mathfrak{F}\mathrm{cd}\, G \leq \mathcal{H}_{\mathfrak{F}}\mathrm{cd}\, G \leq \mathcal{M}_{\mathfrak{F}}\mathrm{cd}\, G \leq \mathrm{cd}_{\mathfrak{F}}\, G$$

Contribution: In [18] I study the Bredon cohomology with coefficients restricted to the family of cohomological Mackey functors, proving that for all groups G, the invariants $\mathcal{H}_{\mathfrak{F}}\mathrm{cd}\, G$ and $\mathfrak{F}\mathrm{cd}\, G$ coincide. I also characterise the Mackey and cohomological Mackey versions of the FP_n conditions of group cohomology. In [19], I prove that for groups G with $\mathfrak{F}\mathrm{cd}\, G$ finite, $\mathfrak{F}\mathrm{cd}\, G = \mathrm{Gcd}\, G$ and provide partial results on the behavious of $\mathfrak{F}\mathrm{cd}$ under group extensions.

Questions: As well as the Kropholler–Mislin conjecture, there are many other problems in this area: There are no known examples where $\mathrm{Gcd}\, G$ and $\mathcal{M}_{\mathfrak{F}}\mathrm{cd}\, G$ differ, the behaviour of $\mathfrak{F}\mathrm{cd}$ and $\mathcal{M}_{\mathfrak{F}}\mathrm{cd}$ under group extensions is not fully understood, and there are questions concerning the precise relationship with other invariants not mentioned above.

2. Poincaré duality groups and generalisations

A *duality group* is a group G of type FP for which

$$(\dagger) \qquad H^i(G, \mathbb{Z}\, G) \cong \begin{cases} \mathbb{Z}\text{-flat} & \text{if } i = n \\ 0 & \text{else.} \end{cases}$$

for some integer n, and is called a *Poincaré duality group* if in addition $H^n(G, \mathbb{Z}\, G) \cong \mathbb{Z}$ [2].

If a group G has cocompact manifold model for $\mathrm{E}G$ then G is a finitely presented Poincaré duality group. The converse, called the PD^{n} conjecture is a significant open question [20, 6]. Bredon–Poincaré duality groups are a generalisation introduced by Davis and Leary, mimicing the condition that G has a cocompact manifold model for $\underline{\mathrm{E}}G$ whose fixed point sets are contractible submanifolds [7, 13].

Contribution: In [17], I study Bredon duality and Bredon–Poincaré groups in detail, giving several sources of examples, characterising them in low dimensions, and looking at their behaviour under taking graphs of groups and direct products.

Questions: The PD^{n} conjecture generalises — does every Bredon–Poincaré duality group with N_H/H finitely presented for all finite subgroups H admit a manifold model for $\underline{E}G$ whose fixed point sets are contractible submanifolds? There are also open questions regarding the Bredon cohomological dimension of Bredon–Poincaré duality groups and their behaviour under extensions.

3. Houghton's Group

Houghton's group H_n was introduced by Houghton in 1978 [11] and Brown later showed that H_n is FP_{n-1} but not FP_n. Degenhardt introduced a generalisation in his thesis, the braided Houghton's group H_n^{br} [8], the construction is similar to the construction of the braided Thompson's groups.

Contribution: At the beginning of my PhD I looked at Bredon cohomological finiteness conditions satisfied by Houghton's groups, and calculated which finiteness conditions are satisfied by centralisers of virtually cyclic subgroups of Houghton's group [16].

Questions: Bux has has conjectured that H_n^{br} is FP_{n-1} but not FP_n as well [4], and it may be possible to solve this conjecture using similar techniques to those used to prove the braided Thompson's group is FP_∞ [5].

References

[1] Abdolnaser Bahlekeh, Fotini Dembegioti, and Olympia Talelli. Gorenstein dimension and proper actions. *Bull. Lond. Math. Soc.*, 41(5):859–871, 2009.

[2] Robert Bieri and Beno Eckmann. Groups with homological duality generalizing Poincaré duality. *Invent. Math.*, 20:103–124, 1973.

[3] Noel Brady, Ian J. Leary, and Brita E. A. Nucinkis. On algebraic and geometric dimensions for groups with torsion. *Journal of the London Mathematical Society*, 64(02):489–500, 2001.

[4] Kai-Uwe Bux. Tangling and braiding the chessboard complex. *preprint*, 2003.

[5] Kai-Uwe Bux, Martin Fluch, Marco Schwandt, Stefan Witzel, and Matthew C. B. Zaremsky. The braided Thompson's groups are of type F_∞. *preprint*, 2012.

[6] Michael W. Davis. The cohomology of a Coxeter group with group ring coefficients. *Duke Math. J.*, 91(2):297–314, 1998.

[7] Michael W. Davis and Ian J. Leary. Some examples of discrete group actions on aspherical manifolds. *High-Dimensional Manifold Topology Proceedings of the School ICTP: School on High-Dimensional Manifold Topology*, pages 139–150, 2003.

[8] F. Degenhardt. *Endlichkeitseigenschaften gewisser Gruppen zon Zöpfen unendlicher Ordnung*. PhD thesis, Frankfurt, 2000.

[9] Dieter Degrijse. Bredon cohomological dimensions for proper actions and Mackey functors. *ArXiv e-prints*, 2013.

[10] Martin J. Dunwoody. Accessibility and groups of cohomological dimension one. *Proc. London Math. Soc.*, s3-38(2):193–215, 1979.

[11] C. H. Houghton. The first cohomology of a group with permutation module coefficients. *Archiv der Mathematik*, 31:254–258, 1978. 10.1007/BF01226445.

[12] Wolfgang Lück and David Meintrup. On the universal space for group actions with compact isotropy. In *Geometry and topology: Aarhus (1998)*, volume 258 of *Contemp. Math.*, pages 293–305. Amer. Math. Soc., Providence, RI, 2000.

[13] Conchita Martinez-Pérez. Euler classes and bredon cohomology for groups with restricted families of finite subgroups. *Mathematische Zeitschrift*, pages 1–20, 2013.

[14] Conchita Martinez-Pérez and Brita E. A. Nucinkis. Cohomological dimension of Mackey functors for infinite groups. *J. London Math. Soc. (2)*, 74(2):379–396, 2006.

[15] Brita E. A. Nucinkis. Cohomology relative to a G-set and finiteness conditions. *Topology and its Applications*, 92(2):153 – 171, 1999.

[16] Simon St. John-Green. Centralisers in Houghton's groups. *preprint, to appear Proc. Edinburgh Math. Soc.*, 2012.

[17] Simon St. John-Green. Bredon–Poincaré duality groups. *preprint*, 2013.

[18] Simon St. John-Green. Cohomological finiteness conditions for Mackey and cohomological Mackey functors. *ArXiv e-prints*, 2013.

[19] Simon St. John-Green. On the Gorenstein and $\mathfrak{F}$-cohomological dimensions. *ArXiv e-prints*, 2013.

[20] C. T. C. Wall, editor. *Homological group theory*, volume 36 of *London Mathematical Society Lecture Note Series*, Cambridge, 1979. Cambridge University Press.