Proper actions and Mackey functors

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March 14, 2014

Models for $\underline{E}G$

Let G be a discrete group.

Definition

A model for $\underline{E}G$ is a G-CW complex X where

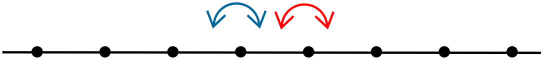
1. G acts properly and cellularly on X.
2. X^H is contractible for all finite subgroups $H \leq G$.

Models for $\underline{E}G$ are unique up to G-homotopy equivalence.

Models for $\underline{E}G$

Examples

1. If G is torsion free, a model for $\underline{E}G$ is the universal cover of a $K(G, 1)$.
2. If G is finite, a model for $\underline{E}G$ is a point.
3. If $G = (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$, the infinite dihedral group, then a model for $\underline{E}G$ is the real line.



Kropholler–Mislin conjecture

Definition

1. $\underline{\mathrm{gd}}\, G$ is the minimal dimension of a model for $\underline{E}G$.
2. n_G is the minimal dimension of a proper contractible G-CW complex.

Clearly $n_G \leq \underline{\mathrm{gd}}\, G$.

Conjecture (Kropholler–Mislin)

$$n_G < \infty \Rightarrow \underline{\mathrm{gd}}\, G < \infty$$

Theorem (Lück)

$n_G < \infty \Rightarrow \underline{\mathrm{gd}}\, G < \infty$ for groups with bounded lengths of chains of finite subgroups.

Bredon cohomology

The algebraic side of models for $\underline{E}G$

Let $\mathcal{O}_{\mathfrak{F}}$ be the category

$$\text{Objects}(\mathcal{O}_{\mathfrak{F}}) = \{G/H : H \text{ a finite subgroup of } G\ \}$$

$$\text{Morphisms}_{\mathcal{O}_{\mathfrak{F}}}(G/H, G/K) = \{G\text{-maps } G/H \to G/K\}$$

A *Bredon module* is a contravariant functor from $\mathcal{O}_{\mathfrak{F}}$ into abelian groups.

- Bredon modules form an abelian category.
- Morphisms between Bredon modules are natural transformations.
- One can define projective Bredon modules.

Bredon cohomology

The algebraic side of models for $\underline{E}G$

One can define

$$H^n_{\mathcal{O}_{\mathfrak{F}}}(G, M) = H^n \operatorname{Hom}_{\text{Bredon}}(P_*, M)$$

where

1. P_* is a projective resolution of the constant Bredon module $\underline{\mathbb{Z}}$,

$$\underline{\mathbb{Z}}(G/H) = \mathbb{Z}$$

$$\underline{\mathbb{Z}}(G/H \xrightarrow{\alpha} G/K) = \mathrm{id}_{\mathbb{Z}}$$

2. M is any Bredon module.

Bredon cohomological dimension

The algebraic side of $\underline{\mathrm{gd}}\, G$

Definition

$$\underline{\mathrm{cd}}\, G = \max\{n : H^n_{\mathcal{O}_{\mathfrak{F}}}(G, M) \neq 0 \text{ for some Bredon module } M\}.$$

Theorem (Dunwoody, Lück–Meintrup)

$\underline{\mathrm{cd}}\, G = \underline{\mathrm{gd}}\, G$ *except possibly* $\underline{\mathrm{cd}}\, G = 2$ *and* $\underline{\mathrm{gd}}\, G = 3$.

Example (Brady–Leary–Nucinkis)

There exist groups with $\underline{\mathrm{cd}}\, G = 2$ and $\underline{\mathrm{gd}}\, G = 3$.

$\underline{\mathrm{FP}}_n$ conditions

The algebraic side of finite-type models for $\underline{E}G$

Definition

A model for $\underline{E}G$ is finite-type if it has finitely many G-orbits of cells in each dimension.

Definition

We say G is $\underline{\mathrm{FP}}_n$ if there is a resolution of projective Bredon modules

$$\cdots \to P_i \to \cdots \to P_0 \to \underline{\mathbb{Z}}$$

where P_i is finitely generated for all $i \leq n$.

Theorem (Lück–Meintrup)

There is a finite-type model for $\underline{E}G$ if and only if G is $\underline{\mathrm{FP}}_\infty$ and $N_G H/H$ is finitely presented for all finite $H \leq G$.

$\mathfrak{F}$-cohomology

The algebraic side of n_G?

A short exact sequence of $\mathbb{Z}G$-modules

$$0 \to A \to B \to C \to 0 \qquad (\star)$$

is $\mathfrak{F}$-split if it splits when restricted to $\mathbb{Z}H$ for any finite subgroup H. A $\mathbb{Z}G$-module P is $\mathfrak{F}$-projective if

$$0 \to \mathrm{Hom}(P, A) \to \mathrm{Hom}(P, B) \to \mathrm{Hom}(P, C) \to 0$$

is exact for all $\mathfrak{F}$-split $(\star)$.

Example

$\mathbb{Z}[G/H]$ is $\mathfrak{F}$-projective for H any finite subgroup of G.

$\mathfrak{F}$-cohomological dimension

The algebraic side of n_G?

Definition (Nucinkis)

The $\mathfrak{F}$-cohomological dimension $\mathfrak{F}\mathrm{cd}\, G$ is the minimum length of an $\mathfrak{F}$-split resolution of $\mathbb{Z}$ by $\mathfrak{F}$-projectives.

Proposition (Bouc, Kropholler–Wall)

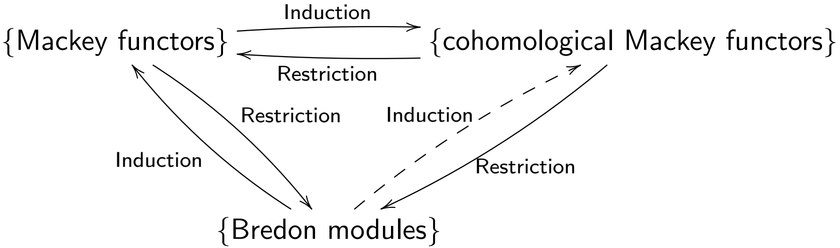
The cellular chain complex of any proper contractible G-CW complex is $\mathfrak{F}$-split.

Corollary

$$\mathfrak{F}\mathrm{cd}\, G \leq n_G$$

We know of no example where $n_G \neq \mathfrak{F}\mathrm{cd}\, G$.

Algebraic Invariants



Mackey functors

Example

1. Representations $\rho : G \to GL_n(\mathbb{R})$.
2. Group (co)homology $H^n(G, M)$ and $H_n(G, M)$.
3. K-Theory of group rings $K_0(\mathbb{Z}\, G)$.
4. Fixed points M^- of a $\mathbb{Z}\, G$-module.

All possess *induction, restriction, and conjugation*.

Mackey functors

A Mackey functor is a map

$$M : \{G/H : H \text{ a finite subgroup of } G\} \to \{\text{abelian groups}\}$$

with maps (for all finite subgroups $K \leq H$ of G and all $g \in G$)

1. $\mathrm{Ind}_K^H : M(G/K) \to M(G/H)$,
2. $\mathrm{Res}_K^H : M(G/H) \to M(G/K)$,
3. $c_g = c_{g,H} : M(G/H^g) \to M(G/H)$.

Such that for all finite subgroups $J \leq K \leq H$

1. $\mathrm{Ind}_H^H = \mathrm{Res}_H^H = c_{h,H} = \mathrm{id}$ for all $h \in H$.
2. $\mathrm{Res}_J^K \circ \mathrm{Res}_K^H = \mathrm{Res}_J^H$ and $\mathrm{Ind}_K^H \circ \mathrm{Ind}_J^K = \mathrm{Ind}_J^H$.
3. $c_g c_h = c_{gh}$ for all $g, h \in G$.
4. $\mathrm{Res}_K^H \circ c_g = c_g \, \mathrm{Res}_{K^g}^{H^g}$ and $\mathrm{Ind}_K^H \circ c_g = c_g \circ \mathrm{Ind}_{K^g}^{H^g}$.
5. $\mathrm{Res}_J^H \circ \mathrm{Ind}_K^H = \sum_{x \in J \backslash H / K} \mathrm{Ind}_{J \cap K^{x^{-1}}}^J \circ c_x \circ \mathrm{Res}_{J^x \cap K}^K$.

Mackey functors as Bredon modules

Let M be a Mackey functor and

$$\begin{aligned} \alpha : G/H &\to G/K \\ H &\mapsto gK \end{aligned}$$

a G-map, then M is a Bredon module by defining $M(\alpha)$ to be the composition

$$M(G/K) \stackrel{\mathrm{Res}^K_{H^g}}{\to} M(G/H^g) \stackrel{c_g}{\to} M(G/H).$$

Definition

$$\underline{\mathrm{cd}}_{\mathrm{Mackey}}\, G = \max\{n : H^n_{\mathcal{O}_{\mathfrak{F}}}(G, M) \neq 0 \text{ for } M \text{ some Mackey functor}\}$$

For all groups G,

$$\underline{\mathrm{cd}}_{\mathrm{Mackey}}\, G \leq \underline{\mathrm{cd}}\, G.$$

$\mathfrak{F}\mathrm{cd}\, G \leq \underline{\mathrm{cd}}_{\mathrm{Mackey}}\, G$

Theorem (Martínez-Pérez–Nucinkis)

$$\mathfrak{F}\mathrm{cd}\, G = \max\{n : H^n_{\mathcal{O}_{\mathfrak{F}}}(G, M^-) \neq 0\}$$

where M^- is the fixed point functor of some $\mathbb{Z}G$-module, ie.

$$M^-(G/H) = M^H.$$

Corollary

$$\mathfrak{F}\mathrm{cd}\, G \leq \underline{\mathrm{cd}}_{\mathrm{Mackey}}\, G$$

Mackey cohomological dimension

Theorem (Martínez-Pérez–Nucinkis)

For any group G and Mackey functor M

$$H^n_{\mathcal{O}_{\mathfrak{F}}}(G, M) \quad \Big(= H^* \operatorname{Hom}_{Bredon}(P_*, M)\Big) \\ = H^* \operatorname{Hom}_{Mackey}(Q_*, M)$$

where Q_* is a projective resolution of the Burnside functor B^G by Mackey functors.

The proof is by inducing a projective resolution of Bredon modules to a projective resolution of Mackey functors.

Theorem (Martínez-Pérez–Nucinkis)

For any virtually torsion-free group G,

$$\mathfrak{F}\mathrm{cd}\, G = \underline{\mathrm{cd}}_{Mackey}\, G = \mathrm{vcd}\, G.$$

Cohomological Mackey functors

A Mackey functor is cohomological if for all finite subgroups $K \leq H$ of G

$$\mathrm{Ind}_K^H \circ \mathrm{Res}_K^H = (m \mapsto |H : K| m).$$

Example

1. Group cohomology $H^n(G, M)$.
2. Fixed point functors M^-, for any $\mathbb{Z}G$-module M.

Cohomological Mackey functors

Yoshida's description

Yoshida showed cohomological Mackey functors can be described as contravariant functors

$$\mathcal{H}_{\mathfrak{F}} \to \{\text{abelian groups}\}$$

where

$$\text{Objects}(\mathcal{H}_{\mathfrak{F}}) = \{G/H : \; H \text{ a finite subgroup of } G\}$$

$$\text{Morphisms}_{\mathcal{H}_{\mathfrak{F}}}(G/H, G/K) = \text{Hom}_{\mathbb{Z}G}(\mathbb{Z}[G/H], \mathbb{Z}[G/K])$$

If $K \leq H$ then Res^H_K corresponds to projection

$$\mathbb{Z}[G/K] \to \mathbb{Z}[G/H]$$

and Ind^H_K corresponds to

$$\mathbb{Z}[G/H] \to \mathbb{Z}[G/K]$$
$$H \mapsto \sum_{h \in H/K} hK$$

Cohomological Mackey functors

Free modules

- Cohomological Mackey functors form an abelian category.
- Morphisms between cohomological Mackey functors are natural transformations.

Example

If H is a finite subgroup of G, the cohomological Mackey functor $\mathbb{Z}[G/H]^-$ is free.

Cohomological Mackey functors

Theorem (S)

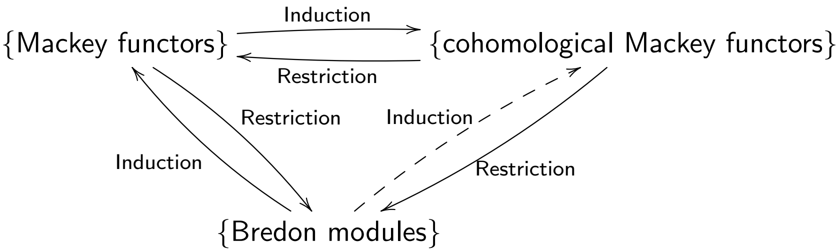
For any group G and cohomological Mackey functor M

$$H^n_{\mathcal{O}_{\mathfrak{F}}}(G, M) \quad \Big(= H^* \operatorname{Hom}_{Bredon}(P_*, M)\Big)$$
$$= H^* \operatorname{Hom}_{coMack}(Q_*, M)$$

where Q_* is a projective resolution of the fixed point functor $\mathbb{Z}^-$ by cohomological Mackey functors.

The proof is via inducing a projective Bredon module resolution of $\underline{\mathbb{Z}}$ to a projective cohomological Mackey functor resolution of $\mathbb{Z}^-$.

Interactions



Cohomological Mackey functors

$$\underline{\mathrm{cd}}_{\mathrm{coMack}}\, G = \max\{n : H^n_{\mathcal{O}_{\mathfrak{F}}}(G, M) \neq 0 \text{ for } M \text{ some cohomological Mackey functor.}\}$$

Or, $\underline{\mathrm{cd}}_{\mathrm{coMack}}\, G$ is the minimum length of a projective resolution of $\mathbb{Z}^-$ by cohomological Mackey functors.

Theorem (S)

$$\mathfrak{F}\mathrm{cd}\, G = \underline{\mathrm{cd}}_{coMack}\, G$$

Proof idea:

1. Evaluating a free cohomological Mackey functor $\mathbb{Z}[G/H]^-$ at $G/1$ gives an $\mathfrak{F}$-projective module.
2. To see $\mathfrak{F}\mathrm{cd}\, G \leq \underline{\mathrm{cd}}_{\mathrm{coMack}}\, G$ we must check that evaluating at $G/1$ gives an $\mathfrak{F}$-split resolution.
3. For the other direction we use a result of Gandini, giving an $\mathfrak{F}$-split resolution of $\mathbb{Z}$ by modules of the form $\mathbb{Z}[G/H]$.

Questions

$$\mathfrak{F}\mathrm{cd}\, G = \underline{\mathrm{cd}}_{\mathrm{coMack}}\, G \leq \underline{\mathrm{cd}}_{\mathrm{Mackey}}\, G \leq \underline{\mathrm{cd}}\, G \leq \underline{\mathrm{gd}}\, G$$

Theorem (Degrijse)

For groups with bounded lengths of chains of finite subgroups,

$$\underline{\mathrm{cd}}_{Mackey}\, G = \underline{\mathrm{cd}}_{coMack}\, G.$$

Question

1. Is $\underline{\mathrm{cd}}_{\mathrm{Mackey}}\, G = \underline{\mathrm{cd}}_{\mathrm{coMack}}\, G$?
2. Are n_G and $\underline{\mathrm{cd}}_{\mathrm{Mackey}}\, G$ connected?

Group Extensions

Let

$$1 \to N \to G \to Q \to 1 \qquad (\star)$$

be a short exact sequence of groups.

Proposition (S)

If $\underline{\mathrm{cd}}_{\mathrm{coMack}}\, G < \infty$ then

$$\underline{\mathrm{cd}}_{\mathrm{coMack}}\, G \leq \underline{\mathrm{cd}}_{\mathrm{coMack}}\, N + \underline{\mathrm{cd}}_{\mathrm{coMack}}\, Q.$$

Theorem (Degrijse)

If there is a group extension $(\star)$ for which $\underline{\mathrm{cd}}_{\mathit{coMack}}$ fails to be sub-additive then there is a group extension $(\star)$ with N a finite group for which it fails to be subadditive.

$\underline{\mathrm{FP}}_n$ conditions

Definition

1. G is Mackey-$\underline{\mathrm{FP}}_n$ if there is a projective resolution P_* of B^G by Mackey functors with P_i finitely generated for all $i \leq n$.
2. G is coMack-$\underline{\mathrm{FP}}_n$ if there is a projective resolution P_* of $\mathbb{Z}^-$ by cohomological Mackey functors with P_i finitely generated for all $i \leq n$.

Theorem (S)

1. *G is Mackey-$\underline{\mathrm{FP}}_n$ if and only if G is $\underline{\mathrm{FP}}_n$.*
2. *G is coMack-$\underline{\mathrm{FP}}_n$ if and only if G is $\mathfrak{F}\mathrm{FP}_n$.*

Questions

If G acts properly on a contractible G-CW complex with finitely many orbits of cells in each dimension then G is $\mathfrak{F}\mathrm{FP}_\infty$, equivalently coMack-$\underline{\mathrm{FP}}_\infty$.

Question

If G is coMack-$\underline{\mathrm{FP}}_\infty$ then does G act properly on a contractible G-CW complex with finitely many orbits of cells in each dimension?