Techniques for Calculating the Hausdorff Dimension of Fractal Sets Generated by Iterated Function Schemes

Simon St. John-Green, University of Warwick

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0 Introduction

As objects to be studied mathematically, fractals defy definition. Falconer[1] calls a set F a fractal if it satisfies the following conditions:

1. F has a fine structure.
2. F is too irregular (both locally and globally) to be defined in regular geometric language.
3. F usually contains some self-similarity.
4. F is usually defined in a simple fashion (eg. recursively).

These sorts of objects arise often both in nature and in other areas of mathematics. Many attractors of dynamical systems for example have fractal properties. We are interested here in the size or dimension of fractal sets. The idea of dimension is similarly hard to define, but roughly speaking we want to associate a real number $\dim F$ to our fractal sets which measures the 'space' taken up by F in small sets around its points. We can write down a short wish list of properties that would be useful for any notion of dimension:

1. $\dim F = n$ if F is an n-dimensional manifold.
2. $\dim F_1 \leq \dim F_2$ if $F_1 \subseteq F_2$.
3. The dimension is unaffected by adding or subtracting 'small' sets, for example countable sets of points in $\mathbb{R}^n$.

The *topological dimension* (or *Lebesgue covering dimension*) may seem like a good start. It is defined by the minimal value of n such that every finite open cover has a finite sub-cover with no point contained in more that $n+1$ elements. While it does satisfy these properties it does not really capture the whole picture. For example consider the cantor set $\mathcal{C}$, obtained recursively by starting with the unit interval $[0,1]$ and removing the middle third to give $\mathcal{C}_1$, then removing the middle third of the two intervals that are left to give $\mathcal{C}_2$ and so on. If we calculate the topological dimension of this we find it is 0 (this follows since the cantor set is totally disconnected). However it is clearly not a 'small' set, for a start it is uncountable. Trying to use Lebesgue measure as a starting point is similarly unhelpful, the Lebesgue measure of the cantor set is also 0. The notion

Figure 1: The Cantor Set

of dimension most commonly used is that of Hausdorff dimension, first thought of by Felix Hausdorff in the 1930s. He built on Carathéodory's ideas about constructing measures. The properties of Hausdorff dimension were then expanded on by his students, most notably Bescovitch. Hausdorff dimension satisfies the properties on our wishlist, as will be shown in more detail when it is defined later. The main problem with Hausdorff dimension is it can be fairly hard to calculate in general, most of the techniques discussed here involve placing restrictions on the construction of F in order to obtain useful estimates. Some other notions of dimension have been developed which are easier to calculate, such as box-counting dimension, packing dimension and Fourier dimension but these mostly do not satisfy the third wish in our wish list, and such are much less useful. A discussion of these can however be found in Falconer[1].
This report will start with some foundational material, starting with a couple of methods of constructing measures, moving on to the definition of Hausdorff dimension, a couple of basic methods of estimating it and some more interesting methods that can be used to estimate invariant sets of collections of functions (iterated function systems). Some techniques for implicitly calculating the Hausdorff dimension will be discussed and finally a quick look at classification of quasi-circles, a class of fractals which arise naturally as the Julia sets of some quadratic functions. The reader should skip any initial sections which are already familiar, as results from these can always be referred back to later.

1 Foundations

Much of the later material will rely on constructing measures in the two following ways and as the Hausdorff dimension also follows from these constructions it seems a natural place to start.

1.1 Measures and Pre-Measures on Metric Spaces

I will give two methods to construct a measure from a pre-measure in a general metric space, both of which will come in useful later on, when we need to work with both sequence spaces and with $\mathbb{R}^n$. These are called

Method I and Method II by Rogers[8]. (X, d) will always denote a general metric space, and the diameter of $U \subset X$ with respect to d will be denoted by $|U| := \sup_{x,y \in U} d(x, y)$.

Definition 1.1. *If (X, d) is a metric space, say a function τ on some class $\mathcal{C}$ of subsets of X is a pre-measure, if*

1. $\emptyset \in \mathcal{C}$ *and* $\tau(\emptyset) = 0$
2. $0 \leq \tau(C) \leq +\infty$ *for all* $C \in \mathcal{C}$

Here I will say that a set function $\mu : X \to \mathbb{R}_{\geq 0}$ is a *countably sub-additive measure* if it satisfies all the usual axioms, including $\mu(\cup_{i=0}^{\infty} F_i) \leq \sum_{i=0}^{\infty} \mu(F_i)$. This function is called a *measure* (or *countably additive measure*) if this inequality can be improved to equality. The reader should recall the standard result from measure theory that for any countably sub-additive measure μ, the set of μ-measurable sets in X form a σ-algebra, and that μ restricted to this σ-algebra is a (countably additive) measure.

Lemma 1.2. *METHOD I*
If τ is a pre-measure on some class $\mathcal{C}$ of subsets of X, then the following is a countably sub-additive measure on X:

$$\mu(F) = \inf \left\{ \sum_{i=0}^{\infty} \tau(U_i) : U_i \in \mathcal{C} \ , \ F \subseteq \cup_{i=0}^{\infty} U_i \right\}$$

Proof. (1) $0 \leq \tau(C) \leq \infty \, \forall C \in \mathcal{C}$, so clearly $0 \leq \mu(F) \leq \infty \, \forall E \subseteq X$
(2) $\mu(\emptyset) = 0$ is clear as the empty set has diameter zero.
(3) If $F_1 \subseteq F_2$, then any cover of F_2 is a cover of F_1, giving $\mu(F_1) \leq \mu(F_2)$.
(4) It remains to prove for some collection $\{F_i\}_{i=0}^{\infty}$, we have $\mu(\cup_{i=0}^{\infty} F_i) \leq \sum_{i=0}^{\infty} \mu(F_i)$. The result is clearly trivial if $\sum_{i=0}^{\infty} \mu(F_i) = \infty$, so assume it is finite, and hence that $\mu(F_i)$ is finite for each i. Now given some $\varepsilon > 0$, choose a cover $\{C_j^i\}_{j=0}^{\infty}$ of F_i with sets from $\mathcal{C}$, such that $\sum_{j=0}^{\infty} \tau(C_j^i) \leq \mu(F_i) + \varepsilon \cdot 2^{-i}$. Now,

$$\mu\left(\bigcup_{i=0}^{\infty} F_i \right) \leq \sum_{i,j} \tau(C_j^i) \leq \sum_{i=0}^{\infty} \mu(F_i) + \varepsilon \cdot 2^{-i} = \sum_{i=0}^{\infty} (F_i) + 2\varepsilon$$

As ε was arbitrary, this gives the required result.

□

Unfortunately, Borel sets are not in general measurable with respect to measures from Method I constructions. Since this is such a useful property most of the measures we use will be constructed using the method below, which is in some sense a refinement of Method I.
Method II restricts the covers allowed to those with diameter at most δ. We will take a family of such measures (one for each δ), and then take the supremum. Since it is easy to construct a measure using a supremum over a family of measures in generality, I will give this as a lemma.

Lemma 1.3. *If $\{\mu_i\}_{i \in I}$ is a family of countably sub-additive measures on X over some indexing set I, then $\mu(F) = \sup_{i \in I} \mu_i(F)$ is a countably sub-additive measure on X*

Proof. (1) $0 \leq \mu(F) \leq \infty \, \forall F \subseteq X$ and $\mu(\emptyset) = 0$ are clear.
(2) If $F_1 \subseteq F_2 \subseteq X$, then $\mu(F_1) = \sup_{i \in I} \mu_i(F_1) \leq \sup_{i \in I} \mu_i(F_2) = \mu(F_2)$
(3) If $\{F_j\}_{j=0}^{\infty}$ are subsets of X, then:

$$\mu\left(\bigcup_{j=0}^{\infty} F_j \right) = \sup_{i \in I} \mu_i \left(\bigcup_{j=0}^{\infty} F_j \right) \leq \sup_{i \in I} \sum_{j=0}^{\infty} \mu_i(F_j) \leq \sum_{j=0}^{\infty} \mu(F_j) = \sum_{j=0}^{\infty} \mu(F_j)$$

□

Lemma 1.4. *METHOD II*
If τ is a pre-measure on $\mathcal{C}$, then $\mu(F) = \sup_{\delta > 0} \mu_\delta(F)$ is an outer measure on X, where

$$\mu_\delta(F) = \inf \left\{ \sum_{i=0}^{\infty} \tau(U_i) : U_i \in \mathcal{C} \ , \ F \subseteq \bigcup_{i=0}^{\infty} U_i \ , \ |U_i| < \delta \ \forall i \right\}$$

Proof. Write $\mathcal{C}_\delta = \{U \in \mathcal{C} \ : \ |U| < \delta\}$, and write τ_δ for the restriction of τ to $\mathcal{C}_\delta$, it is clear that τ_δ is a pre-measure. Now $\mu_\delta(F) = \inf\{\sum_{i=0}^{\infty} \tau_\delta(U_i) \ : \ U_i \in \mathcal{C}_\delta \ , \ F \subset \cup_{i=0}^{\infty} U_i\}$, which is a countably sub-additive measure by Method I (Lemma 1.2), and μ is a countably sub-additive measure by Lemma 1.3. □

Notice that as δ decreases, the set of admissible δ covers decreases, and hence μ_δ increases. So we could replace the definition of μ by $\mu(F) = \lim_{\delta\to 0} \mu_\delta(F)$. Measures constructed in this fashion are often called *measures of Hausdorff type.*
Recall the Borel Sets of a space are the sets in the minimal σ-algebra in X which contains all the open sets of X. A countably sub-additive measure on a metric space (X, d) is called a countably sub-additive metric measure if $d(A, B) > 0$ implies $\mu(A \cup B) = \mu(A) + \mu(B)$ and one can prove that if μ is a countably sub-additive metric measure, then all Borel sets are measurable (see Munroe[20] Corollary 13.2.1). The following facts will be useful later on, but they are not particularly illuminating to prove. The proofs can be found in Rogers[8] or Munroe[20].

Lemma 1.5. *For a measure constructed via Method II.*

1. $\mu\left(\bigcup_{i=0}^{\infty} U_i\right) = \sup_i \mu(U_i)$
2. *μ is a metric measure, and hence all Borel Sets are μ-measureable.*

Definition 1.6. *Throughout this report, the support of a measure $\mu : X \to \mathbb{R}$ refers to the smallest closed set S with $\mu(X \setminus A) = 0$. So a point x is in the support if and only if $\mu(B(x, r)) > 0$ for all r. Moreover when we speak of a measure on a set $F \subseteq X$ we mean a measure with support contained in F.*

1.2 Constructing Hausdorff Dimension

The previous section leads nicely into the definition of Hausdorff dimension.

Definition 1.7. *We define the s-dimensional Hausdorff Measure for some $F \subset X$ of a general metric space (X, d) as*

$$\mathcal{H}^s(F) = \lim_{\delta\to 0} \inf \left\{ \sum_{i=0}^{n} |U_i|^s : F \subseteq \bigcup_{i=1}^{n} U_i, 0 < |U_i| < \delta \right\}$$

and the Hausdorff dimension

$$\dim_{\mathcal{H}} = \sup\{x : \mathcal{H}^s(x) = \infty\} = \inf\{x : \mathcal{H}^s(x) = 0\}$$

To see that $\mathcal{H}^s$ is a measure, it suffices to notice that the set function $U \to |U|^s$ is a pre-measure on the set of open sets, and the measure is that constructed via the Method II construction in 1.4. To see $\dim_{\mathcal{H}}$ is well defined, if $t > s$, and $\{U_i\}$ is a δ-cover of F, then:

$$\sum_i |U_i|^t \le \sum_i |U_i|^{t-s}|U_i|^s \le \delta^{t-s} \sum_i |U_i|^s$$

So, taking infima, we see that if $\mathcal{H}^s(F) < \infty$, then $\mathcal{H}^t(F) = 0$. So there is a critical value of s at which $\mathcal{H}^s(F)$ jumps from ∞ to 0, which is exactly $\dim_{\mathcal{H}}$.
The Hausdorff dimension generalises well the idea of dimension of a set, for example the Hausdorff dimension of a smooth m-dimensional manifold in $\mathbb{R}^n$ is m, and for any set $F \subset \mathbb{R}^n$, with $\dim_{\mathcal{H}}(F) < 1$, the set F is a totally disconnected set of points. A proof of these facts can be found in Falconer[1]. Connections between Hausdorff dimension and Lebesgue measure are discussed in depth in Rogers[8].

Example 1.8. *The canonical example of a fractal set is the cantor set, described by taking the unit interval and removing the middle third, then repeating the process on the remaining two intervals, and so on. $\mathcal{C} = \cap_{i=0}^{\infty} C_i$, where $C_0 = [0, 1]$, $C_1 = [0, 1/3] \cup [2/3, 1]$. and so on. We can calculate an upper bound for the Hausdorff dimension of this by taking the intervals making up the sets C_i as an open 3^{-k} cover of $\mathcal{C}$. Then $\Sigma|C_i|^s = 2^k 3^{-ks}$ and setting $s = \log 2/\log 3$ gives $\mathcal{H}^s(\mathcal{C}) \le 1$, and hence $\dim_{\mathcal{H}} \mathcal{C} \le s$.*
In fact, $\dim_{\mathcal{H}} \mathcal{C} = \log 2/\log 3$ but the upper bound is more complicated to calculate. Although it can be done from here the calculation isn't that constructive, so it is left until section 1.4.

Now that we have a notion of dimension, the obvious next question is which maps leave the Hausdorff dimension invariant.

Lemma 1.9. *If $f : F \to X$ is a map with $d(f(x), f(y)) \le cd(x, y)^\alpha$, for all $x, y \in F$, with some constants $c > 0$ and $\alpha > 0$ (This is called the* Hölder Condition of Exponent *α). Then for any s, $\mathcal{H}^{s/\alpha}(f(F)) \le c^{s/\alpha}\mathcal{H}^s(F)$.*

To see this let $\{U_i\}$ be a δ-cover of F, then since $|f(F \cap U_i)| \le c|F \cap U_i|^\alpha \le c|U_i|^\alpha$, it follows that $\{f(F \cap U_i)\}$ is a $c\delta^\alpha$-cover of $f(F)$. So $\sum_i |f(F \cap U_i)|^{s/\alpha} \le c^{s/\alpha} \sum_i |U_i|$, and taking infima and letting $\delta \to 0$ gives the required result. From here, the following lemma is clear.

Lemma 1.10. *If $f : F \to X$ satisfies the Hölder condition of exponent α, then $\dim_{\mathcal{H}} f(F) \le 1/\alpha \dim_{\mathcal{H}} F$. In particular, if f is a bi-Lipschitz mapping, then $\dim_{\mathcal{H}} f(F) = \dim_{\mathcal{H}} F$.*

Finally, one more lemma that will come in useful later and follows immediately since the projection map is Lipschitz.

Lemma 1.11. *If* proj *is projection from* $\mathbb{R}^n$ *onto the last* m *coordinates, then* $\dim_{\mathcal{H}}(\operatorname{proj} F) \leq \min\{\dim_{\mathcal{H}} F, 1\}$ *for any* $F \subseteq \mathbb{R}^n$.

1.3 Box-Counting Dimension

The box-counting dimension will be needed in passing in some of the later sections, so a brief exposition of it is given here. In fact, box-counting dimension has been around since before Hausdorff dimension but it is significantly less useful than Hausdorff dimension, as will be discussed at the end of this section.

Definition 1.12. *Given a non-empty bounded subset* $F \subset \mathbb{R}^n$ *let* $N_\delta(F)$ *be the smallest number of sets of diameter at most* δ *needed to cover* F*. Then we define the upper and lower box counting dimension (*$\overline{\dim}_B$ *and* $\underline{\dim}_B$ *respectively) as*

$$\overline{\dim}_B(F) = \limsup_{\delta\to 0} \frac{\log N_\delta(F)}{-\log\delta} \quad \textit{and} \quad \underline{\dim}_B(F) = \liminf_{\delta\to 0} \frac{\log N_\delta(F)}{-\log\delta}$$

If these two limits are equal we define the box counting dimension $\dim_B$ *as* $\dim_B(F) = \overline{\dim}_B(F) = \underline{\dim}_B(F)$.

This definition gives the idea behind the box counting dimension, but in practice we usually use the following alternative definition to make it easier to calculate. Say a collection of the form

$$\Big\{[m_1\delta, \delta(m_1+1)\delta] \times \cdots \times [m_n\delta, (m_n+1)\delta] : m_1, \ldots, m_n \in \mathbb{Z}\Big\}$$

is a δ-mesh, and define $N'_\delta(F)$ to be the number of cubes in a δ-mesh which intersects F. It is clear that $N_{\delta\sqrt{n}}(F) \leq N'_\delta(F)$, so for δ small enough

$$\frac{\log N_{\delta\sqrt{n}}(F)}{-\log(\delta\sqrt{n})} \leq \frac{N'_\delta(F)}{-\log\sqrt{n} - \log\delta}$$

So

$$\underline{\dim}_B F \leq \liminf_{\delta\to 0} \frac{\log N'_\delta(F)}{-\log\delta} \text{ and } \overline{\dim}_B \leq \limsup_{\delta\to 0} \frac{\log N'_\delta(F)}{-\log\delta}$$

To get the opposite inequalities in the above it suffices to notice that any set of diameter at most δ is contained in at most 3^n mesh cubes, so $N'_\delta(F) \leq 3^n N_\delta(F)$ and the argument follows along the same lines. Hence we can replace N_δ by N'_δ in the definition of box-counting dimension. It can also be shown that is enough to consider limits along a decreasing sequence $\delta_k = c^k$ for some $c \in (0,1)$. This alternative definition lends itself to empirical calculations, and indeed when people talk about the dimension of a coastline or some other natural phenomenon they will often be talking about an estimation based on the box-counting dimension.
There is yet another alternative definition which deserves a mention here since it will be used in later sections. If $\tilde{N}_\delta(F)$ is the maximum number of disjoint balls of radius δ with centers in F, then

$$\overline{\dim}_B(F) = \limsup_{\delta\to 0} \frac{\log \tilde{N}_\delta(F)}{-\log\delta} \text{ and } \underline{\dim}_B(F) = \liminf_{\delta\to 0} \frac{\log \tilde{N}_\delta(F)}{-\log\delta}$$

This can be easily proved in a similar way to the calculation with N'_δ.

Example 1.13. *The box counting dimension of the cantor set* $\mathcal{C}$ *may be calculated by taking the obvious* δ*-meshes for* $\delta_k = 1/3^k$ *to find* $N'_{\delta_k}(\mathcal{C}) = 2^k$ *and thus*

$$\underline{\dim}_B(\mathcal{C}) = \overline{\dim}_B(\mathcal{C}) = \dim_B(\mathcal{C}) = \lim_{k\to\infty} \frac{\log 2^k}{\log 3^k} = \frac{\log 2}{\log 3}$$

We can relate the box-counting dimension to the Hausdorff dimension in the following way

Lemma 1.14. *For any non-empty bounded* $F \subseteq \mathbb{R}^n$

$$\dim_{\mathcal{H}} F \leq \underline{\dim}_B F \leq \overline{\dim}_B F$$

Proof. The latter inequality is obvious, for the former notice that if F is covered by $N_\delta(F)$ sets of diameter δ then

$$\inf\left\{\sum_{i=0}^{\infty} |U_i|^s \ : \ \{U_i\}_{i=1}^{\infty} \text{ a } \delta\text{-cover of } F\right\} \leq N_\delta(F)\delta^s$$

so if $\mathcal{H}^s(F) > 1$, we have $-\delta^s < N_\delta(F)$ so $s \leq \liminf_{\delta\to 0} \log N_\delta(F)/-\log\delta$, which gives the required result. ☐

The following properties demonstrate some of the other similarities between Hausdorff and box counting dimension, as well as leading us into its main problem.

Lemma 1.15. *1. If F is a smooth n-dim manifold, then $\dim_B F = n$.*
2. If $F_1 \subseteq F_2$ then $\overline{\dim}_B(F_1) \leq \overline{\dim}_B(F_2)$ and $\underline{\dim}(F_1) \leq \underline{\dim}(F_2)$.
3. $\underline{\dim}_B$ and $\overline{\dim}_B$ are bi-Lipschitz invariant.
4. $\underline{\dim}_B(F) = \underline{\dim}_B(\overline{F})$ and $\overline{\dim}_B(F) = \overline{\dim}_B(\overline{F})$, where $\overline{F}$ is the closure of F.

All of these follow from the definitions, but we will only prove the last one here.

Proof. If B_i is a finite collection of sets of diameter at most δ with $F \subset \cup_i B_i$ then $\overline{F} \subset \cup_i \overline{B_i}$ also. □

The last property of the previous lemma is a real problem for the box-counting dimension, as it implies that $\dim_B(\mathbb{Q} \cap [0,1]) = \dim_B([0,1]) = 1$. So the box counting dimension can be affected by countable sets which is something we really don't want (it was wish 3 in our wish list in the introduction). Several people have come up with modified versions of the box-counting definition to remove this issue, for example by taking

$$\dim(F) = \inf\left\{ \sup_{1 \leq i \leq \infty} (\dim_B F_i) \ : \ F \subseteq \bigcup_{i=1}^{\infty} F_i \right\}$$

Unfortunately these types of modifications mean that this new modified dimension is no longer easy to calculate.

1.4 Mass Distribution Principle

The Hausdorff dimension is only going to be of any use if there are ways to calculate it, or at least obtain some bound on its value. Most of the time finding an upper bound on the dimension is easier than finding a lower bound. To find an upper bound it is usually sufficient to compute $\sum |U_i|^s$ for some collection of δ-covers, as it was in the cantor set example. However to find a lower bound it is necessary to consider all δ-covers, which is often much harder. These next two subsections give some useful techniques for finding this lower bound.
The Mass distribution Principle provides a way of estimating the lower bound of the Hausdorff Dimension, roughly speaking if we can create a mass distribution on F which assigns 'less' mass than the 'size' $|U|^s$ of the set U, then we will gain information about $\mathcal{H}^s$.

Lemma 1.16. *Mass Distribution Principle Let μ be a mass distribution on $F \subset \mathbb{R}^n$ (Here we take a* mass distribution *to mean a countably sub-additive measure with $0 < \mu(F) < \infty$), and suppose for some $s \geq 0$, there exists $c > 0$ and $\varepsilon > 0$ such that $\mu(U) \leq c|U|^s$ for all sets U with $|U| \leq \varepsilon$. Then $\mathcal{H}^s(F) \geq \mu(F)/c$ and $s \leq \dim_{\mathcal{H}} F$.*

Proof. Let $\{U_i\}$ be a δ-cover of F, with $\delta \leq \varepsilon$ then

$$0 < \mu(F) \leq \mu\left(\bigcup_{i=1}^{\infty} U_i\right) \leq \sum_{i=1}^{\infty} \mu(U_i) \leq c \sum_{i=1}^{\infty} |U_i|^s$$

Thus, taking infima over all δ-covers gives $\mathcal{H}^s(F) \geq \mu(F)/c > 0$, and so $\dim_{\mathcal{H}}(F) \geq s$. □

Notice there was no requirement in the above for the sets U_i or the set F to be measurable.

Example 1.17. *We can use the mass distribution principle to get an upper bound on the dimension of the cantor set. Recalling the cantor set $\mathcal{C}$ is described by $\mathcal{C} = \cap_{i=0}^{\infty} C_i$, where $C_0 = [0,1]$, and $C_1 = [0,1/3] \cup [2/3,1]$ and so on (defined iteratively by removing the middle third from each interval).*
A (countably sub-additive) measure μ can be defined by letting $\mu(U) = 2^{-k}$ for each interval U in C_k, and then using method I (Lemma 1.2) to generalise this onto arbitrary subsets of $\mathbb{R}$. Now, for any open set $U \subset [0,1]$, let k be the integer with $3^{-k+1} \leq |U| < 3^{-k}$. U can intersect at most one interval in C_k, and hence $\mu(U) \leq 2^{-k}$ and

$$\mu(U) \leq 2^{-k} = 3^{-k\frac{\log 2}{\log 3}} = \left(3^{-k}\right)^{\frac{\log 2}{\log 3}} \leq (3|U|)^{\frac{\log 2}{\log 3}}$$

Hence, by the mass distribution principle, $\frac{\log 2}{\log 3} \leq \dim_{\mathcal{H}}(\mathcal{C})$

1.5 Thermodynamic Formalism

Although the Mass Distribution Principle is useful it does require the estimation of $\mu(U)$ for a large number of small sets U. This method replaces that requirement with the calculation of a single integral.

Definition 1.18. *For $s \geq 0$, define the s-energy $I_s(\mu)$ of μ as:*

$$I_s(\mu) = \int \phi_s(x)d\mu(x) \quad \textit{where} \quad \phi_s(x) = \int \frac{d\mu(y)}{|x-y|^s}$$

First we need a preliminary lemma:

Lemma 1.19. *Let μ be a mass distribution on $\mathbb{R}^n$, $F \subset \mathbb{R}^n$, and $0 < c < \infty$ a constant.*

$$\textit{If } \limsup_{r\to 0} \frac{\mu(B(x,r))}{r^s} < c \, \forall x \in F \text{ , } \textit{then } \mathcal{H}^s(F) \geq \frac{\mu(F)}{c}$$

Proof. For each $\delta > 0$, define the set $F_\delta = \{x \in F \ : \ \mu(B(x,r)) < cr^s \, \forall \, 0 < r \leq \delta\}$. Now, any δ-cover U_i is necessarily a δ-cover of F_δ, and for any $x \in F_\delta$, $U_i \subseteq B(x, |U_i|)$ (recall a δ-cover has sets of *diameter* at most δ) Now, $\mu(U_i) \leq \mu(B) \leq c|U_i|^s$, and $\mu(F_\delta) \leq \sum_i \{\mu(U_i) \ : \ U_i \cap F_\delta \neq \emptyset\} \leq c \sum_i |U_i|^s$, and so $\mu(F_\delta) \leq c\mathcal{H}^s(F)$. Finally, as F_δ are increasing as δ decreases, $\lim_{\delta\to 0} \mu(F_\delta) = \mu(\cup_{\delta>0} F_\delta) = \mu(F)$ (by continuity of μ), giving $\mu(F) \leq c\mathcal{H}^s(F)$. □

Now we are ready to prove the main result, that given a mass distribution on F with finite s-energy, we can conclude that the Hausdorff dimension of F is greater than s. This can be a very powerful way of obtaining a lower bound on the Hausdorff dimension.

Lemma 1.20.
1. *Let $F \subset \mathbb{R}^n$, if there is a mass distribution μ on F and $I_s(\mu) < \infty$, then $\mathcal{H}^s(F) = \infty$ and $\dim_{\mathcal{H}} F \geq s$.*
2. *If F is a Borel set with $\mathcal{H}^s(F) > 0$ then there exists a mass distribution μ on F with $I_t(\mu) < \infty$ for all $0 < t < s$.*

Only the first part of this lemma will be proved, since this is the most useful for later on, the second part is just stated for interest. The proof of the first part given here is an elaboration of that from Falconer[1] and a proof of the second part can be found there also.

Proof. Let μ be a mass distribution with support (recall Definition 1.6) contained in F and $I_s(\mu) < \infty$. Then define

$$G = \left\{ x \in F : \limsup_{r\to 0} \frac{\mu(B(x,r))}{r^s} > 0 \right\}$$

. For any $x \in G$ we can find some $\varepsilon > 0$ and sequence $r_i \to 0$ such that $\mu(B(x,r_i)) \geq \varepsilon r_i^s$. Now $\mu(\{x\}) = 0$, else $\phi_s(x) = \infty$ and so $I_s(\mu) = \infty$, which cannot happen. By continuity of μ,

$$\lim_{n\to\infty} \mu(B(x,1/n)) = \mu\left(\bigcap_{n=1}^{\infty} B(x,1/n) \right) = 0$$

So we may choose q_i, $0 < q_i < r_i$ such that $\mu(B(x,q_i)) \leq 3/4\varepsilon r_i^s$. Next, define the annuli $A_i = B(x,r_i) \setminus B(x,q_i)$. So $\mu(A_i) \geq 1/4\varepsilon r_i^s$. Passing to a subsequence if necessary, it can be assumed that $r_{i+1} < q_i$ for all i, making the annuli A_i disjoint. So for all $x \in G$

$$\phi_s(x) = \int \frac{d\mu(y)}{|x-y|^s} \geq \sum_{i=1}^{\infty} \int_{A_i} \frac{d\mu(y)}{|x-y|^s} \geq \frac{1}{r_i^{-s}} \int_{A_i} d\mu(y) \geq \sum_{i=1}^{\infty} \frac{1}{4} \varepsilon r_i^s r_i^{-s} = \infty$$

But $I_s(\mu) = \int \phi_s(x)d\mu(x) < \infty$ by assumption, so $\phi_s(x) < \infty$ for μ-almost all x, and hence $\mu(G) = 0$.
Now, for all $x \in F \setminus G$, it is the case that $\limsup_{r\to 0} \frac{\mu(B(x,r))}{r^s} = 0$ so by lemma 1.19, we have for all $c > 0$

$$\mathcal{H}^s(F) \geq \mathcal{H}^s(F \setminus G) \geq \mu(F \setminus G)/c \geq (\mu(F) - \mu(G))/c = \mu(F)/c$$

So $\mathcal{H}^s(F) = \infty$ and $\dim_{\mathcal{H}}(F) \geq s$ □

Example 1.21. *Thermodynamic Formalism can be used to give the same lower bound on the Hausdorff dimension of the cantor set $\mathcal{C}$ as was obtained using the mass distribution principle. Let μ be the same measure as was constructed in the previous example (split the mass evenly across each interval at each step of the construction).*

Firstly, notice that $\Gamma = \{(x,y) \in \mathcal{C}^2 \,:\, x = y\}$ *is a* $(\mu \times \mu)$*-null set since*

$$(\mu \times \mu)(\Gamma) = \int_{\mathcal{C}} \mu(\Gamma(y)) d\mu(y) = 0$$

where $\Gamma(z) = \Gamma \cap \{(x,y) \,:\, y = z\}$*. Now we can split up* $\mathcal{C} \times \mathcal{C}$ *in the following way. Write* $\mathcal{C}_j^i$ *for the* i^{th}*interval from the left in the* j^{th}*iterate of the construction and let* $U_1 = \mathcal{C}_1^2 \times \mathcal{C}_1^1 \cup \mathcal{C}_2^1 \times \mathcal{C}_1^1$*,* $U_k = \left(\cup_{i\neq j} C_k^i \times C_k^j\right) \setminus U_{k-1}$ *for* $k \geq 2$*. So* U_k *is made up of* 2^k *disjoint sets and* $\mu(U_k) = 2^k \cdot 4^{-k} = 2^{-k}$*. Now for any* $s < \log 2/\log 3$*.*

$$\int_{\mathcal{C}\times\mathcal{C}} \frac{1}{|x-y|^s} d(\mu \times \mu) = \sum_{k=1}^{\infty} \int_{U_k} \frac{d(\mu\times\mu)}{|x-y|^s} + \int_{\Gamma} \frac{d(\mu\times\mu)}{|x-y|^s} = \sum_{k=1}^{\infty} \mu(U_k) 3^{ks} \leq \sum_{j=1}^{\infty} 3^{ks} 2^{-s} < \infty$$

Since for all $(x,y) \in U_k$*,* $|x-y| \geq 1/3^k$*. Now by Fubini's theorem (see [20] Theorem 29.7)*

$$\int_{\mathcal{C}} \int_{\mathcal{C}} \frac{d\mu(x)d\mu(y)}{|x-y|^s} = \int_{\mathcal{C}\times\mathcal{C}} \frac{d(\mu\times\mu)}{|x-y|^s} < \infty$$

Which gives the lower bound $\dim_{\mathcal{H}} \mathcal{C} \geq s$ *for all* $s < \log 2/\log 3$ *by Lemma 1.20 and hence* $\dim_{\mathcal{H}} \mathcal{C} \geq \log 2/\log 3$*.*

1.6 Sequence Spaces

A small detour has to be taken here to discuss sequence spaces, as much of the later material is described most easily using them. Direct comparisons between the dimensions of the sequence space and some attractors of dynamical systems can be made, and the sequence space will be useful for creating measures to be used in conjunction with the mass distribution principle and thermodynamic formalism.
First, some notation. If $\mathcal{A}$ is a finite set, then write $\mathcal{A}^k$ for the set of length-k sequences over $\mathcal{A}$, $\mathcal{A}^F = \cup_{i=0}^{\infty} \mathcal{A}^k$ for the set of all finite sequences and $\mathcal{A}^{\mathbb{N}}$ for the set of one-sided infinite sequences over $\mathcal{A}$. In this report, $\mathcal{A}$ will always be the set $\{1, \ldots, m\}$, for some $m \in \mathbb{N}$.
For $\mathbf{i} = (i_0, \ldots, i_n) \in \mathcal{A}^F$ and $\mathbf{j} = (j_0, j_1, \ldots) \in \mathcal{A}^F$ or $\mathcal{A}^{\mathbb{N}}$, write $\mathbf{i}, \mathbf{j}$ for the sequence $(i_0, \ldots i_n, j_0, j_1, \ldots)$. Also, write $\mathbf{i} < \mathbf{j}$, if $\mathbf{j} = \mathbf{i}, \mathbf{i\prime}$, and $|\mathbf{i}|$ for the length of the sequence. If $\mathbf{i}, \mathbf{j} \in \mathcal{A}^{\mathbb{N}}$, then we denote by $\mathbf{i} \wedge \mathbf{j}$ the maximal subsequence such that $\mathbf{i} \wedge \mathbf{j} < \mathbf{i}$ and $\mathbf{i} \wedge \mathbf{j} < \mathbf{j}$. We denote by $[i_0, \ldots, i_n] = \{\mathbf{i} \,:\, (i_0, \ldots i_n) < \mathbf{i}\}$ the cylinder sets and finally, $\sigma : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$ denotes the usual shift on $\mathcal{A}^{\mathbb{N}}$, taking $(i_0, i_1, \ldots) \mapsto (i_1, i_2, \ldots)$. Next we state some basic properties of $\mathcal{A}^{\mathbb{N}}$, which will come in useful later when we want to describe measures on the space.

Lemma 1.22. *(1) The topology* $\mathcal{B}$ *generated by taking the cylinder sets as a basis forms a* σ*-algebra in* $\mathcal{A}^{\mathbb{N}}$*.*
(2) The function $d : \mathcal{A}^{\mathbb{N}} \times \mathcal{A}^{\mathbb{N}} \to \mathbb{R}_{\geq 0}$*, given by* $d(\mathbf{i}, \mathbf{j}) = 2^{-|\mathbf{i}\wedge\mathbf{j}|}$*, is a metric on* $\mathcal{A}^{\mathbb{N}}$*.*
(3) The topology $\mathcal{B}$ *is also the Borel* σ*-algebra generated by the metric* d*.*
(4) $\mathcal{A}^{\mathbb{N}}$ *is compact with respect to the above topology.*

Proof. (1) The complement of a cylinder set is a union of cylinder sets.
(2) For any $\mathbf{k} \in \mathcal{A}^{\mathbb{N}}$, $d(\mathbf{i}, \mathbf{j}) \leq 2^{-\min\{|\mathbf{i}\wedge\mathbf{k}|, |\mathbf{k}\wedge\mathbf{j}|\}} = \max\{d(\mathbf{i}, \mathbf{k}), d(\mathbf{k}, \mathbf{j})\}$
(3) The open balls with respect to d are exactly the cylinder sets.
(4) Using the metric d, $\mathcal{A}^{\mathbb{N}}$ can be shown to be compact, by showing it to be complete and totally bounded. For total boundedness, it suffices to notice that any sequence is a distance of at most 2 away from any other. To show completeness, take a Cauchy sequence in $\mathbf{i}_n \in \mathcal{A}^{\mathcal{N}}$, and notice the Cauchy condition is equivalent to

$$\forall C \in \mathbb{N}, \exists N_C \in \mathbb{N} \text{ such that } \forall n, m > N_C \text{ we have that } |\mathbf{i}_m \wedge \mathbf{i}_n| > C$$

So every sequence in the set $I_C = \{\mathbf{i}_n : n > N_C\}$ agrees on the first C terms. Then a sequence $\mathbf{i}$ can be chosen by taking the Cth element of $\mathbf{i}$ as the Cth element of any sequence in I_C, and then this sequence is the limit of $\mathbf{i}_n$.

□

A new metric ρ can be defined on $\mathcal{A}^{\mathbb{N}}$, by setting $\rho(\mathbf{i}, \mathbf{j}) = c_{i_0} \ldots c_{i_n}$, where $n = |\mathbf{i} \wedge \mathbf{j}|$ and $\{c_0, \ldots, c_m\}$ is a set of integers with $0 < c_i < 1$. From now on, this metric will be referred to as the metric generated by $\{c_1, \ldots, c_m\}$. Interestingly, ρ is in fact an ultrametric, it satisfies $\rho(x,y) \leq \max\{(\rho(x), \rho(y)\}$ and it will come in useful when relating the Hausdorff dimension of subsets of the shift space to the Hausdorff dimension of certain fractals.

Lemma 1.23. *The Borel sigma algebra* $\mathcal{B}$ *generated by* d *is the same as the Borel* σ*-algebra generated by* ρ*.*

Proof. The open balls with respect to ρ are exactly the cylinder sets. □

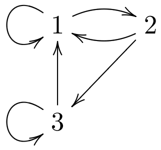
Most of the content in the coming sections will use countably sub-additive measures defined on the σ-algebra of all subsets of a space and we need not worry about the above. However, there are one or two places where it is important to have a (countably additive) measure defined on the Borel σ-algebra (for example in section 3 where we need to use Fubini's theorem).

1.7 Subshifts of Finite Type

In this section , $\mathcal{A} = \{1, \dots, m\}$. If $K \subseteq \mathcal{A}^{\mathbb{N}}$ we say K is a *subshift* if $\sigma(K) \subseteq K$. If $B = b_{i,j}$ is an $m \times m$ matrix with entries in $\{0, 1\}$, then the subset $\mathcal{A}_B$ of $\mathcal{A}^{\mathbb{N}}$ can be defined by allowing only the elements $\mathbf{i} = (i_0, i_1, \dots)$ with the property that b_{i_j, i_j+1} for all $j \in \mathbb{N}$. This subset is called the subshift of finite type associated to B, and B is known as the transition matrix. Clearly, $\mathcal{A}_B$ satisfies $\sigma(\mathcal{A}_B) \subseteq \mathcal{A}_B$. Notice that if B has a row i which contains all zeros, then no $\mathbf{i} \in \mathcal{A}_B$ may contain b_i (since there would be no choices for b_{i+1}. Since we are only interested in modeling infinite sequence spaces, we will exclude matrices which contain entirely zeros rows. Subshifts of finite type are often described using their associated directed graph G, given by taking the vertices as the points in $\mathcal{A}$ and saying there is an edge from i to j in G if $b_{i,j} = 1$.

Example 1.24. *If $\mathcal{A} = \{1, 2, 3\}$ then the transition matrix B gives associated graph G:*

$$B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$



The metrics d and ρ can be restricted to $\mathcal{A}_B$, and will still be written d and ρ, they retain all the properties of Lemma 1.22.

1.8 Iterated Function Systems

Definition 1.25. 1. *A mapping $S : D \to D$ is called a* contraction *on some closed $D \subset \mathbb{R}^n$ if there is some c, with $0 < c < 1$ and $|S(x) - S(y)| \leq c|x - y|$ for all $x, y \in D$. If $|S(x) - S(y)| = c|x - y|$, then S is a* contracting similarity *(or just* similarity*). The constant c is usually called the* contraction factor*.*

2. *A finite family $\{S_i\}_{i=1}^m$ of contractions is called an* iterated function system*, and we call a non-empty subset $\Lambda \subset D$, such that $\Lambda = \cup_{i=1}^m S_i(\Lambda)$, the* invariant set*.*

3. *A set Λ which can be represented as the invariant set of an iterated function system is called a* self-similar set*.*

Iterated Function Schemes can be modeled using sequence spaces and subshifts of sequence spaces. If $\{S_1, \dots S_m\}$ is an IFS then let $\mathcal{A} = \{1, \dots, m\}$, then choose some compact bounded set $B \subset \mathbb{R}^n$ big enough that $B \subseteq S_i(B)$ for all i, and define:

$$S(\mathbf{i}) = S_{i_1} \circ S_{i_2} \circ \cdots \circ S_{i_n}(B) \text{ if } \mathbf{i} = (i_1, i_2, \dots, i_n) \in \mathcal{A}^F$$

$$S(\mathbf{i}) = \bigcap_{r=0}^{\infty} S_{i_1} \circ S_{i_2} \circ \cdots \circ S_{i_r}(B) \text{ if } \mathbf{i} = (i_1, i_2, \dots) \in \mathcal{A}^{\mathbb{N}}$$

Now $|\cap_{k=0}^n S_{i_0} \circ S_{i_1} \cdots S_{i_k}(B)| < |B| c_{i_0} \cdots c_{i_n}$, so the diameters are decreasing and hence by Cantor's Intersection Theorem, the intersection $S(\mathbf{i}) = \cap_{k=0}^{\infty} S_{i_0} \circ S_{i_1} \cdots S_{i_k}(B)$ is a single point. If we want to work with some subshift $K \subseteq \mathcal{A}^{\mathbb{N}}$ instead of with the full shift, we can simply restrict the above map to the subshift.
Notice also that with the above definition we could have used any compact bounded set B with the property that $S_i(B) \subset B$ for all i. The next lemma the most important lemma in this section as it shows the invariant set Λ is completely determined by the iterated function system when we are working with the full shift.

Lemma 1.26. *Any Iterated Function Scheme determines a unique invariant set Λ, where Λ is compact and non-empty.*

Proof. Existence: Firstly, $\cup_{\mathbf{i} \in \mathcal{A}^n} S(\mathbf{i})$ are non empty compact sets satisfying $\bigcup_{\mathbf{i} \in \mathcal{A}^n} S(\mathbf{i}) \subset \bigcup_{\mathbf{i} \in \mathcal{A}^{n-1}} S(\mathbf{i})$. Hence we have that

$$\Lambda = \bigcap_{n=1}^{\infty} \left(\bigcup_{\mathbf{i} \in \mathcal{A}^n} S(\mathbf{i}) \right)$$

is non-empty and compact (by Cantor's Intersection theorem again) and satisfies $\Lambda = \cup_{i=1}^m S_i(\Lambda)$.
Uniqueness: Firstly we define the Hausdorff metric

$$d(A, B) = \inf \{\delta \,:\, A \subseteq B_\delta \,,\, B \subseteq A_\delta\} \text{ where } U_\delta = \{x \,:\, |x - u| \leq \delta \text{ for some } u \in U\}$$

It is easy to check this satisfies the conditions of a metric. Next assume $\tilde{\Lambda}$ also satisfies $\tilde{\Lambda} = \cup_{i=1}^m S_i(\tilde{\Lambda})$ and notice

$$d(\Lambda, \tilde{\Lambda}) = d\left(\bigcup_{i=1}^m S_i(\Lambda), \bigcup_{i=1}^m S_i(\tilde{\Lambda}) \right) \leq \max_{1 \leq i \leq m} d\left(S_i(\Lambda), S_i(\tilde{\Lambda}) \right) \leq \max_{1 \leq i \leq m} c_i d(\Lambda, \tilde{\Lambda})$$

So because $0 < c_i < 1$ for all i, $d(\Lambda, \tilde{\Lambda}) = 0$. □

Also since $\cup_i S_i(\cup_{\mathbf{i}\in\mathcal{A}^{\mathbb{N}}} S(\mathbf{i})) = \cup_{\mathbf{i}\in\mathcal{A}^{\mathbb{N}}} S(\mathbf{i})$, by uniqueness $\Lambda = \cup_{\mathbf{i}\in\mathcal{A}^{\mathbb{N}}} S(\mathbf{i})$. We now consider properties of the map S, with respect to the metric ρ on $\mathcal{A}^{\mathbb{N}}$, defined in section 1.6.

Lemma 1.27. *If $S : \mathcal{A}^{\mathbb{N}} \to \Lambda$ as in the previous section.*

1. *The map S is Lipschitz and hence continuous.*
2. *If the iterated function system is made up of contracting similarities, and S is injective then S is bijective and bi-Lipschitz.*

Proof. 1. For $\mathbf{i}, \mathbf{j} \in \mathcal{A}^{\mathbb{N}}$, we have $S(\mathbf{i}), S(\mathbf{j}) \in S_{i_0} \circ S_{i_1} \cdots S_{i_r}(\Lambda)$, where $r = |\mathbf{i} \wedge \mathbf{j}|$. Hence $|S(\mathbf{i}) - S(\mathbf{j})| \leq |\Lambda|\rho(\mathbf{i}, \mathbf{j})$.

2. Since Λ is exactly $\cup_{\mathbf{i}\in\mathcal{A}^{\mathbb{N}}} S(\mathbf{i})$, the map $S : \mathcal{A}^{\mathbb{N}} \to \Lambda$ is always surjective, so if S is injective it is bijective also. S can also be seen to be bi-Lipschitz. Let $\mathbf{i}, \mathbf{j} \in \mathcal{A}^{\mathbb{N}}$ agree up to the first n-1 places and disagree on the nth, so since the S_i are contractions:

$$|S(\mathbf{i}) - S(\mathbf{j})| = c_{i_0} \cdots c_{i_n} \left|S(\tilde{\mathbf{i}}) - S(\tilde{\mathbf{j}})\right| = \rho(\mathbf{i}, \mathbf{j}) \left|S(\tilde{\mathbf{i}}) - S(\tilde{\mathbf{j}})\right| \geq \rho(\mathbf{i}, \mathbf{j}) \min_{i \neq j} \left(d\left(S_i(\Lambda), S_j(\Lambda)\right)\right)$$

where $\tilde{\mathbf{i}}, \tilde{\mathbf{j}} \in \mathcal{A}^{\mathbb{N}}$ differ in the first position, d is the usual distance between two sets, and

$$\min_{i \neq j} \left(d\left(S_i(\Lambda), S_j(\Lambda)\right)\right) > 0$$

because Λ is compact, and hence closed, so if the distance between $S_1(\Lambda)$ and $S_2(\Lambda)$ was zero, they would share a point, which would contradict the injectivity of S.

□

The first part of the above lemma doesn't require the fact that S is defined on the whole shift and is still true is we restrict it to some subshift, the second part of the lemma does require this but if we restrict the map to $S : K \to \cup_{\mathbf{i}\in K} S(\mathbf{i})$ it holds. This sort of technique is discussed in more detail in Section 2.2 on sub-self-similar sets.
This result above can make the map S incredibly useful. If S can be shown to be injective then it is bi-Lipschitz, then by Lemma 1.10 S preserves Hausdorff dimension. So we can say that $\dim_{\mathcal{H}}(K) = \dim_{\mathcal{H}}(\Lambda)$.

Example 1.28. *The Sierpiński Triangle, can be defined as the invariant set Λ by the similarities $\{S_1, S_2, S_3\}$, where*

$$S_1 = T, S_2 = T + \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, S_3 = T + \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ where } T = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

This iterated function system has unique invariant set $\Lambda = \cup_{\mathbf{i}\in\mathcal{A}^{\mathbb{N}}} S(\mathbf{i})$. We write $\mathcal{I}^k$ for the set of cylinder sets of length k, and notice that $\mathcal{I}^k$ is a cover of $\mathcal{A}^{\mathbb{N}}$ with $|\mathcal{I}^k| = 2^k$ and $|I| = 3^{-k}$ for all $I \in \mathcal{I}^k$. Hence if $s = \log 3/\log 2$ then $\sum_{I\in\mathcal{I}^k} |I|^s = 3^k 2^{-ks} = 1$. Hence $\mathcal{H}^s(\Lambda) \leq 1$ and $\dim_{\mathcal{H}} \Lambda \leq s$.

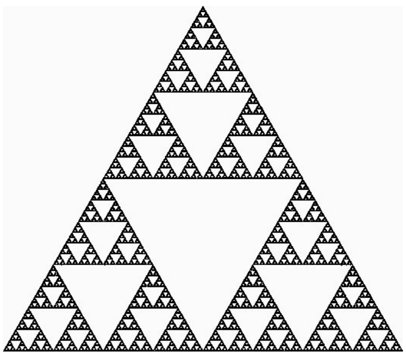


Figure 2: The Sierpiński Triangle

Unfortunately, in practice S is not always injective as a map on $\mathcal{A}^{\mathbb{N}}$. To get around this we can restrict the order in which we apply the similarities, ie. model this as as a subshift.

Example 1.29. *If we restrict the order in which we can apply the similarities when constructing the Sierpiński triangle, we can create a modified Sierpiński triangle. For example, allowing all compositions except S_1 with itself can be modeled with a subshift of finite type generated by the transition matrix*

$$B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Now there is the problem that it is unclear how to calculate an upper bound for $\dim_{\mathcal{H}}$*. A technique for doing this will be given in section 2.3.*

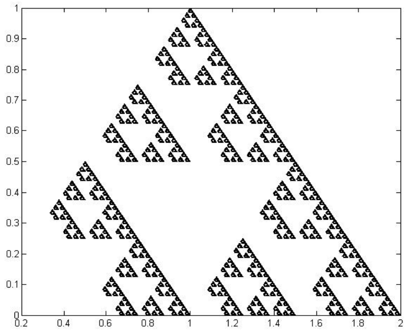


Figure 3: Modified Sierpinski Triangle

Although calculations of the dimensions of these two Sierpiński carpets could be done from here using the same method used to find the dimension of the cantor set, it is best left until later when more techniques have been developed.

1.9 Measures on Sequence Spaces

An alternative use of sequence spaces in modeling Iterated Function Systems comes from the fact that it is in general much easier to define a measure on $\mathcal{A}^{\mathbb{N}}$, than it is on Λ. The measure defined on $\mathcal{A}^{\mathbb{N}}$ can then be used to define a measure on some invariant set Λ of an iterated function system, as demonstrated in the following lemma.

Lemma 1.30. *1. If μ is a countable sub-additive measure on $\mathcal{A}^{\mathbb{N}}$, then ν is a countably sub-additive measure on Λ, where $\nu(U) = \mu\{\mathbf{i} : S(\mathbf{i}) \in U\}$. Moreover if μ is a (countably sub-additive) mass distribution on $\mathcal{A}^{\mathbb{N}}$ then ν is a (countable sub-additive) mass distribution on Λ.*

2. If μ is a (countably additive) measure on $\mathcal{B}$, the Borel σ-algebra on $\mathcal{A}^{\mathbb{N}}$, then ν is a (countably additive) measure on the Borel σ-algebra of $\mathbb{R}^n$.

Proof. 1. $\nu(\emptyset) = 0$ and $U \subseteq V \subseteq \mathcal{A}^{\mathbb{N}} \Rightarrow \nu(U) = \mu(S^{-1}(U)) \subseteq \mu(S^{-1}(V)) = \nu(V)$ follow easily. Moreover, given U_i a countable collection of sets

$$\nu\left(\bigcup_{i=0}^{\infty} U_i\right) = \mu\left(\bigcup_{i=0}^{\infty} S^{-1}(U_i)\right) \leq \sum_{i=0}^{\infty} \nu(U_i)$$

If μ is a mass distribution, ie $0 < \mu(\mathcal{A}^{\mathbb{N}}) < \infty$, then $\nu(\Lambda) = \mu(\mathcal{A}^{\mathbb{N}})$, and hence ν is also. We can write μ as $\nu(U) = \mu(S^{-1}(U))$. Now since S is continuous and continuous functions pull back open sets to open sets and closed sets to closed sets, S also pulls back Borel sets to Borel sets. So if $\{U_i\}_{i=1}^{\infty}$ is a countable collection of disjoint Borel sets in $\mathbb{R}^n$ then $\mu(S^{-1}(\cup_i U_i) = \mu(\cup_i S^{-1}(U_i)) = \sum_i \mu(S^{-1}(U_i))$, since the $S^{-1}(U_i)$ are disjoint also.

□

This result, and constructions very similar to this will turn out to be extremely helpful in many of the theorems that follow. As an example, consider the *Bernoulli Measure* on $\mathcal{A}^{\mathbb{N}}$ is given by $\mu([i_0, i_1, \ldots i_n]) = p_{i_0} p_{i_1} \ldots p_{i_n}$, where $\{p_j : j \in \mathcal{A}\}$ is a probability vector ($\sum_j p_j = 1$). This defines a countably sub-additive measure by the Method 1 construction in Lemma 1.2, taking the pre-measure on the class of cylinder sets. In fact the Bernoulli Measure is a (countably additive) measure on the σ-algebra $\mathcal{B}$ generated by cylinder sets, a fact which follows from Carathéodory's Extension Theorem ([20] Theorem 11.2).

Example 1.31. *This measure can then be projected down onto, for example, the cantor set $\mathcal{C} \subset X$, in this case $\mathcal{A} = 0, 1$ corresponding to the two similarities S_1 and S_2. Let μ be the Bernoulli measure with probability vector $\{1/2, 1/2\}$, and ν the associated measure on X. It is easy to see this defines the same measure as in the earlier example (Mass Distribution Principle), as a cylinder set $[i_0, \dots i_k]$ in $\mathcal{A}^{\mathbb{N}}$ corresponds to an interval in the k^{th} iterate of the construction, C_k.*

2 Self-Similar Sets

The cantor set example in the previous section didn't really show the full power of using iterated function system and sequence spaces. We now give a simple condition that reduces the calculation of the Hausdorff dimension of an invariant set of an IFS to a very simple form. The condition is called the open set condition, and roughly speaking asks for the IFS to satisfy some sort of separation condition, ie. the images of the contractions S_i cannot 'overlap' too much.

Definition 2.1. *We say an IFS satisfies the* Open Set Condition (OSC)*, if there is a bounded non-empty open set $U \subset \mathbb{R}^n$, such that for all i, $S_i(O) \subset O$, and for all $i \neq j$, $S_i(O) \cap S_j(O) = \emptyset$. Such an open set is called a* feasible open set*.*

We require a few preliminaries before the main proof.

Lemma 2.2. *If $U_i \subset \mathbb{R}^n$ is a collection of disjoint open sets, with each U_i containing a ball of radius $a_1 r$ and contained in a ball of radius $a_2 r$. Then any ball B of radius r intersects at most $(1+2a_2)^n a_1^{-n}$ of the closures $\overline{U_i}$*

Proof. Let $B = B(x,r)$, so if $\overline{U}_i \cap B(x,r) \neq \emptyset$ then $\overline{U}_i \subset B(x, 1+2a_2 r)$. If q sets $\overline{U}_i$ intersect with $B(x,r)$, then we sum up the volume of the (disjoint) interior balls of the U_i to give $q(a_1 r)^n \leq (1+2a_2)^n r^n$. □

Lemma 2.3. *Given $\{c_1, \dots, c_m\}$ with $0 < c_j < 1$ for all j, the set function μ on $\mathcal{A}^{\mathbb{N}} = \{1, \dots, m\}^{\mathbb{N}}$, given by $\mu([i_0, \dots, i_k]) = (c_{i_0} \cdots c_{i_k})^s$ defines a countably sub-additive mass distribution on $\mathcal{A}^{\mathbb{N}}$, where s is the unique real number such that $\sum_j c_j^s = 1$.*

Proof. We use the Method I construction (Lemma 1.2) to define the sub-additive measure on $\mathcal{A}^{\mathbb{N}}$ from the pre-measure μ on the class of cylinder sets. Finally, $\mu(\mathcal{A}^{\mathbb{N}}) = \mu(\cup_j [j]) = \sum_j c_j^s = 1$, so μ is a mass distribution. □

Now we come to the main result, the upper bound in the following theorem is fairly easy to derive, but the lower bound is a little involved, although in essence it relies on defining a measure in the sequence space, pushing it down onto Λ and then appealing to the mass distribution principle.

Theorem 2.4. *Given an IFS of similarities $\{S_i\}_{i=0}^m$ satisfying the open set condition, with ratios $0 < c_i < 1$ and invariant set Λ, then $\dim_{\mathcal{H}} \Lambda = \dim_B \Lambda = s$, where s is given by $\sum_{i=0}^m c_i^s = 1$. Moreover, for this s, $0 < \mathcal{H}^s(\Lambda) < \infty$.*

Only the calculation for the Hausdorff Dimension will be done here, the calculation for the box dimension follows along similar lines. It is also a consequence of Theorem 4.3.

Proof. Recall the maps from the sequence space onto Λ could be defined for any closed bounded B large enough that $B \subseteq S_i(B)$ for all i. Since this is true for Λ itself, we will take that as the definition for now.

$$S(\mathbf{i}) = S_{i_1} \circ \cdots \circ S_{i_n}(\Lambda) \text{ for } \mathbf{i} \in \mathcal{A}^F \text{ and } S(\mathbf{i}) = \lim_{n \to \infty} S_{i_1} \circ \cdots \circ S_{i_n}(\Lambda) \text{ for } \mathbf{i} \in \mathcal{A}^{\mathbb{N}}$$

Firstly, for the upper bound notice that $\Lambda = \cup_{\mathbf{i} \in \mathcal{A}^k} S(\mathbf{i})$ is an open cover of Λ, where $|S(\mathbf{i})| \leq (\max_i \{c_i\})^k |\Lambda|$. So given any $\delta > 0$, we may choose k large enough for $|S(\mathbf{i})| \leq \delta$ for all $\mathbf{i} \in \mathcal{A}^k$. Finally it suffices to notice

$$\sum_{\mathbf{i} \in \mathcal{A}^k} |S(\mathbf{i})|^s = \sum_{(i_0, \dots i_k) \in \mathcal{A}^k} (c_{i_1} \cdots c_{i_k})^s |\Lambda|^s = \left(\sum_{i_1 \in \mathcal{A}} c_{i_1}^s \right) \cdots \left(\sum_{i_k \in \mathcal{A}} c_{i_k}^s \right) |\Lambda|^s = |\Lambda|^s$$

This gives $\mathcal{H}^s(\Lambda) \leq |\Lambda|^s$.
For the lower bound let μ be the mass distribution of the previous lemma, and ν the natural mass distribution on Λ (as in Lemma 1.30) defined by taking $\nu(U) = \mu(\{\mathbf{i} : S(\mathbf{i}) \in U\})$. We show ν satisfies the conditions of the mass distribution principle (lemma 1.16), and $\nu(\Lambda) = 1$. Let O be a feasible open set (as in Definition 2.1) and notice that by the argument of Lemma 1.26 that $\left(\sum_{i=1}^m S_i(\overline{O}) \right)^k$ converges to Λ as $k \to \infty$. Moreover, $S_{i_0} \circ \cdots \circ S_{i_k}(\overline{O}) \supseteq S_{i_0} \circ \cdots \circ S_{i_k}(\Lambda)$ for all $(i_0, \dots, i_k) \in \mathcal{A}^F$. Let B be a ball of radius $r < 1$, we will want to estimate $\nu(B)$.

If we truncate each sequence $\mathbf{i} \in \mathcal{A}^{\mathbb{N}}$ to $i_0, \dots i_k$ where k is the smallest integer such that

$$r \min_{1\le i\le m} c_i \le c_{i_0} \cdots c_{i_k} \le r$$

and denote the set of such sequences by $\mathcal{Q}$. Now, $S_a(O)$ and $S_b(O)$ are disjoint for all $a \neq b$ so $S_{i_0}\circ\cdots\circ S_{i_k}\circ S_a(O)$ and $S_{i_0}\circ\cdots\circ S_{i_k}\circ S_b(O)$ are disjoint for all $a \neq b$ and all $(i_0, \dots, i_k) \in \mathcal{A}^F$. Hence we can conclude

$$\left\{S_{i_0}\circ\cdots\circ S_{i_k}(O) \,:\, (i_0,\dots,i_k)\in\mathcal{Q}\right\} \text{ is a pairwise disjoint collection} \tag{2.1}$$

Notice also that $\Lambda \subset \cup_{\mathcal{Q}} S_{i_0}\circ\cdots\circ S_{i_k}(\Lambda) \subset \cup_{\mathcal{Q}} S_{i_0}\circ\cdots\circ S_{i_k}(\overline{O})$. Choose a_1, a_2 such that O contains a ball of radius a_1 and is contained in a ball of radius a_2. So for all $(i_0, \dots i_k) \in \mathcal{Q}$, $S_{i_0}\circ\cdots\circ S_{i_k}(O)$ is contains a ball of radius $a_1 r \min_i c_i$ and is contained in a ball of radius $c_{i_1}\cdots c_{i_k} a_2$ and hence in a ball of radius $a_2 r$.
Let $\mathcal{Q}_1$ denote the sequences $(i_0, \dots, i_k) \in \mathcal{Q}$ such that $B \cap S_{i_1}\circ\cdots\circ S_{i_k}(\overline{O}) \neq \emptyset$. By (2.1) and Lemma 2.2, B intersects at most $q = (1+2a_2)^n a_1^{-n}(\min_c c_i)^{-n}$ elements of $\mathcal{Q}$. Hence

$$\nu(B) = \mu\{\mathbf{i} \,:\, S(\mathbf{i}) \in B)\} \le \mu \bigcup_{(i_0,\dots,i_k)\in\mathcal{Q}_1} [i_0, \dots i_k] \le \sum_{\mathcal{Q}_1} \mu([i_0,\dots i_k]) = \sum_{\mathcal{Q}_1} (c_{i_0}\cdots c_{i_k})^s \le \sum_{\mathcal{Q}_1} r^s \le r^s q$$

So since U is contained in a ball of radius $|U|$, $\nu(B) \le |U|^s q$ and by the mass distribution principle (Lemma 1.16), $\mathcal{H}^s(\Lambda) \ge 1/q > 0$ and $\dim_{\mathcal{H}} \Lambda \ge s$. □

Example 2.5. *This theorem immediately gives us the dimension of the cantor set $\mathcal{C}$, using the similarities $S_1(x) = x/3$, $S_2(x) = x/3 + 2/3$ defined earlier. $\{S_1, S_2\}$ satisfies the open set condition with the open set $(0,1)$, and hence theorem 2.4 gives $2\left(\frac{1}{3}\right)^s = 1$, where $s = dim_{\mathcal{H}}\mathcal{C}$. Hence* $\dim_{\mathcal{H}}(\mathcal{C}) = \log 2/\log 3$.

Example 2.6. *The Hausdorff dimension of the Sierpiński Triangle Λ given by the similarities $\{S_1, S_2, S_3\}$ is also easily calculated, where*

$$S_1 = T, S_2 = T + \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, S_3 = T + \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ where } T = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

Each S_i clearly has contraction factor $1/2$, and the open set condition holds with open set $(0,2)\times(0,1) \subset \mathbb{R}^2$. So theorem 2.4 gives $\dim_{\mathcal{H}}(\Lambda) = \log 3/\log 2$

2.1 The Open Set Condition

The main condition needed in the previous section was the open set condition (OSC). It is interesting to notice the exact interplay between different conditions such as the open set condition. The results below are given just for interest and will not be used in the remaining material. In theorem 2.4 we showed that $OSC \Rightarrow H^\alpha(F) > 0 \Rightarrow \dim_{\mathcal{H}}(F) = \alpha$, where $\alpha = \dim_B(F)$. It was shown by A. Shief[9] that in fact

$$OSC \Leftrightarrow H^\alpha > 0 \Rightarrow \dim_{\mathcal{H}}(F) = \alpha$$

The final inequality unfortunately does not go both ways as can be illustrated by an example originally due to Mattila. Showing that for any fractal set it is possible to have $\mathcal{H}^1 = 0$ with $\dim_{\mathcal{H}} = \dim_B = 1$ is easy (consider a cantor set where the size of the intervals removed at each stage decreases), but it is significantly harder to find a self-similar example.

Example 2.7. *Consider*

$$T_1 = T_2 = T_3 = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$$

$$a_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \; a_2 = \begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix}, \; a_3 = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{2\sqrt{3}} \end{pmatrix}$$

This creates a variant of the Sierpiński triangle, which satisfies the open set condition with $(0,1)\times(0,\frac{1}{2})$ and using theorem 2.4 it is easy to show $\alpha = \dim_B(\Lambda) = \dim_{\mathcal{H}}(\Lambda) = \log 3/\log 2$. Next, it is possible to say that for almost any line through the origin in $\mathbb{R}^2$ the projection of Λ onto this subspace is self-similar, satisfies the OSC and has Hausdorff and box dimension 1, but $\mathcal{H}^1$ is zero. Unfortunately this requires some tricky theorems involving densities, it in fact follows from Falconer[1] Theorems 5.2, 6.8.

It has also been shown by Bandt and Graf[10] that the OSC is equivalent to what they call the Neighbour Map Condition. The set of neighbour maps is $\mathcal{N} = \{S_{\mathbf{i}}^{-1}\circ S_{\mathbf{j}} : \mathbf{i}, \mathbf{j} \in \mathcal{A}^F \;,\; i_0 \neq j_0\}$ The neighbour map condition then states that there is a constant $\kappa > 0$ such that for all $h \in \mathcal{N}$, $\|h - \mathrm{id}\,\| > \kappa$. The norm here is the usual supremum norm on $\mathbb{R}^n$.

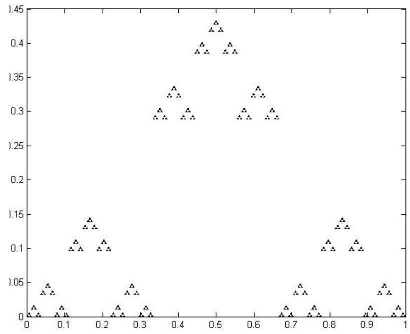


Figure 4: The variant of the Sierpiński triangle from Example 2.7

Finally C. Bandt, N. Viet Hung and H. Rao[11] built on this to give a constructive approach to creating this open set by showing that if

$$V = \{x \,:\, d(x,\Lambda) < d(x,H)\} \text{ where } H = \bigcup_{h\in\mathcal{N}} h(\Lambda)$$

Then V is a feasible open set if OSC holds, and moreover if OSC does not hold then $V = \emptyset$.

2.2 Sub-Self-Similar Sets

In section 2 we found an easy way to calculate the Hausdorff dimension of a space which can be modeled using a full shift $\mathcal{A}^{\mathbb{N}}$ and which satisfies the open set condition. The following result relaxes the first criterion, requiring only that it is modeled by some subshift of the shift space. Throughout this section the metric on $\mathcal{A}^{\mathbb{N}}$ is $d(\mathbf{i},\mathbf{j}) = 2^{-|\mathbf{i}\wedge\mathbf{j}|}$ (this metric was discussed in section 1.6). We will say that Λ is *sub-self-similar* with respect to an iterated function systems $\{S_1,\dots,S_n\}$ if $\Lambda \subseteq \cup_{i=0}^m S_i(\Lambda)$. The next Lemma is very powerful, as it allows us characterise sub-self-similar sets generated by iterated function systems in $\mathbb{R}^n$ using subshifts in $\mathcal{A}^{\mathbb{N}}$.

Example 2.8. *If Λ is a self-similar set, ie. $\Lambda = \cup_i S_i(\Lambda)$ for some iterated function systems $S_1,\dots,S_m$, then $\partial\Lambda \in \Lambda$ since Λ is closed. Let $z \in \partial\Lambda$ with $z \in S_i(\Lambda)$ for some i, then if $z = S_i(x)$ for some $x \in \Lambda^\circ$ then there would be some open set $U \subset I^\circ$ with $S_i(U) \subseteq \Lambda^\circ$ contradicting $z \in \partial\Lambda$. Hence $z \in S_j(\partial\Lambda)$ and so $\partial\Lambda \subseteq \cup_i S_i(\partial\Lambda)$ is a sub-self similar set.*

For example, taking the self-similar set generated by $S_i(x) = T(x) + a_i$ where $T = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$ and $a_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $a_2 = \begin{pmatrix} \frac{1}{3} \\ 0 \end{pmatrix}$, $a_3 = \begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix}$, $a_4 = \begin{pmatrix} 0 \\ \frac{1}{3} \end{pmatrix}$, $a_5 = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$, $a_6 = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$, $a_7 = \begin{pmatrix} 0 \\ \frac{2}{3} \end{pmatrix}$, $a_8 = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$, $a_9 = \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$, $a_{10} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. By the remarks above, $\partial\Lambda$ is a sub-self-similar set with respect to the same $S_1,\dots,S_{10}$.

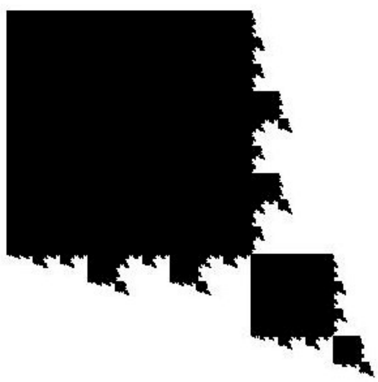


Figure 5: Self-Similar Set of Example 2.8. The Hausdorff dimension can be calculated as being almost certainly $\dim_{\mathcal{H}} \Lambda = \log 10/\log 3$ via Theorem 3.9 in the section on Self-Affine sets (unfortunately, we cannot use Theorem 2.4 since the open set condition does not hold).

Lemma 2.9. *Let $\{S_1,\dots,S_m\}$ be contractions. Then Λ is compact and sub-self-similar for $\{S_1,...,S_m\}$ if and only if $\Lambda = S(K)$ for some compact set $K \subseteq \mathcal{A}^{\mathbb{N}}$ satisfying the condition $\sigma(K) \subseteq K$*

Proof. If K is some compact subset of $\mathcal{A}^{\mathbb{N}}$, with $\sigma(K) \subseteq K$. Then for any $x \in S(K)$, it is true that $x = S(\mathbf{i})$ for some $\mathbf{i} = (i_1, i_2, \ldots) \in K$. Hence $x = S_{i_1}(S(\sigma(\mathbf{i}))) \in S_{i_1}(S(K))$. This shows that $S(K)$ satisfies the Sub-Self-Similar Condition. For the converse, suppose Λ satisfies $\Lambda \subseteq \cup_{i=0}^{m} S_i(\Lambda)$ and let

$$K = \{(i_0, i_1, \ldots) \,:\, S(i_k, i_{k+1}, \ldots) \in \Lambda \,\forall k \in \mathbb{N}\}$$

Clearly $\sigma(K) \subseteq K$. Now, since S is continuous (by lemma 1.27) and Λ is closed we have that $S^{-1}(\Lambda)$ is closed. By continuity of σ,

$$\sigma^{-k}\left(S^{-1}(\Lambda)\right) = \left\{(i_0, i_1, \ldots) \,:\, (i_k, i_{k+1}, \ldots) \in S^{-1}(\Lambda)\right\}$$

is closed. Hence

$$K = \bigcap_{k=1}^{\infty} \left\{(i_0, i_1, \ldots) \,:\, (i_k, i_{k+1}, \ldots) \in S^{-1}(\Lambda)\right\}$$

is closed and compact (recall that $\mathcal{A}^{\mathbb{N}}$ is compact).
It is also clear that $S(K) \subseteq \Lambda$. Now, if $x_0 \in \Lambda$ then by the sub-self-similar condition, there exists an i_0 such that $x_0 = S_{i_0}(x_1)$ for some $x_1 \in \Lambda$. Now, there exists an i_1 such that $x_1 = S_{i_1}(x_2) \in \Lambda$ and so on. Hence $x_0 = \bigcap_{k=0}^{\infty} S_{i_0} \circ \cdots \circ S_{i_k}(B) = S(\mathbf{i})$ for some $\mathbf{i} = (i_0, i_1, \ldots) \in S^{-1}(\Lambda)$ and indeed for any $k \in \mathbb{N}$

$$S(i_k, i_{k+1}, \ldots) = \bigcap_{j=k}^{\infty} S_{i_k} \circ \cdots \circ S_{i_j}(B) = x_k \in \Lambda$$

Hence $x_0 \in S(K)$ and $\Lambda \subseteq S(K)$. $\square$

Example 2.10. *Consider a simplification of Example 2.8, where we take the first nine contractions $S_1, \ldots, S_9$. It is clear that the invariant set of these contractions is simply the unit square $[0,1]^2$. We can then describe the subshift generating $\partial[0,1]^2$ using the graph of figure 6. K_9 represents the complete graph on 9 vertices, and the allowable sequences are those which eventually follow one of the dotted lines into one of the four subgraphs which each give one side of the unit square.*

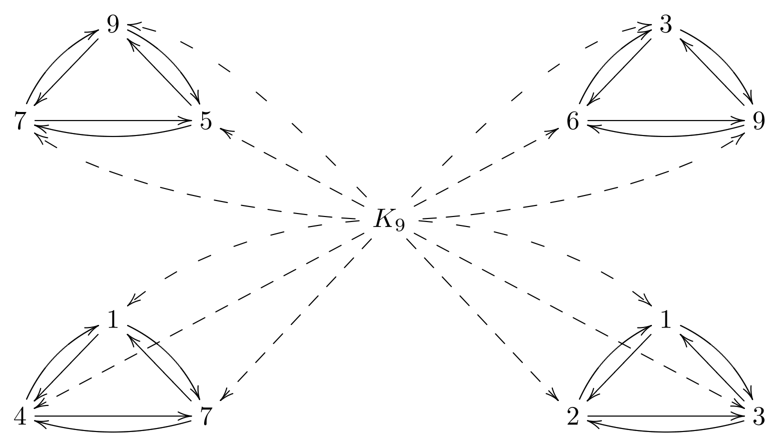


Figure 6: Directed graph showing allowable sequences to generate $\partial[0,1]^2$

Firstly, some notation

1. K^k is the set of sequences in k-length sequences in K and $K^F = \cup_{k=1}^{\infty} K^k$ is the set of finite length sequences in K.
2. If $\{c_1, \ldots c_m\}$ are the contraction ratios for $\{S_1, \ldots, S_m\}$ then we write $c_{\mathbf{i}} = c_{i_0} c_{i_1} \cdots c_{i_n}$ for any $\mathbf{i} = \{i_0, \ldots, i_n\}$. So $c_{\mathbf{i}}$ is the ratio of the contraction $S_{i_0} \circ \cdots \circ S_{i_n}$.

Notice that because of the condition $\sigma(K) \subset K$ we have that if $\mathbf{i} \in \mathcal{A}^k$, $\mathbf{j} \in \mathcal{A}^l$ and $\mathbf{ij} \in K^{k+l}$ then $\mathbf{i} \in K^k$ and $\mathbf{j} \in K^l$. Much of the following material will rely on this simple observation and the following simple Lemma.

Lemma 2.11. *A* sub-multiplicative sequence *is a sequence that satisfies $a_{k+m} \leq a_k a_m$ and for any such sequence the limit $\lim_{k\to\infty}(a_k)^{1/k}$ exists and is equal to $\inf_k (a_k)^{1/k}$.*

Proof. Given $n \in \mathbb{N}$, we can write any $k \in \mathbb{Z}$ as $k = qn + p$ where $q, p \in \mathbb{Z}$ and $0 \leq p < n$. Let $k \geq n$ so by applying the above q times we get $a_k \leq a_n^q a_p$. Hence

$$(a_k)^{1/k} \leq a_n^{q/k} a_p^{1/k} \leq \left((a_n)^{1/n}\right)^{qn/k} a_p^{1/k}$$

Since there are only finitely many choices for a_p, as $k \to \infty$ we get $\overline{\lim}_{k\to\infty}(a_k)^{1/k} \le (a_n)^{1/n}$. Since n was arbitrary we get $\overline{\lim}_{k\to\infty}(a_k)^{1/k} \le \inf_n a_n^{1/n}$. Hence $\overline{\lim}_{k\to\infty}(a_k)^{1/k} \le \inf_n a_n^{1/n} \le \underline{\lim}_{k\to\infty}(a_k)^{1/k}$ which gives the result. □

The next section is done is slightly more generality than may seem necessary, but this means that the material is reusable when we get to the Section 3. Let $\xi : K^F \times \mathbb{R}_{\ge 0} \to \mathbb{R}$ be a function that is

1. Sub-multiplicative in $\mathbf{i}$
2. Strictly decreasing and continuous in s.
3. There exist constants $0 < c_+, c_- < 1$ such that $\xi(\mathbf{i}, s)c_-^{kh} \le \xi(\mathbf{i}, s+h) \le \xi(\mathbf{i}, s)c_+^{kh}$, where $k = |\mathbf{i}|$.
4. $\xi(\mathbf{i}, 0) = 1$ for all $\mathbf{i}$

This represents a generalisation of the function $\mathbf{i}, s \mapsto c_{\mathbf{i}}^s$.

Lemma 2.12. *1. The limit $\tau_\xi(s)$ defined as*

$$\tau_\xi(s) = \lim_{k\to\infty} \left(\sum_{\mathbf{i}\in K^k} \xi(\mathbf{i}, s) \right)^{\frac{1}{k}}$$

exists with $0 \le \tau_\xi(s) < \infty$ and $\sum_{\mathbf{i}\in K^k} \xi(\mathbf{i}, s) \ge \tau_\xi(s)^k$ for all $k \in \mathbb{N}$.

2. There exists a unique $s \ge 0$ such that $\tau_\xi(s) = 1$.

Proof. Submultiplicativity implies that the limit $\tau_\xi(s)$ exists and also that $\sum_{\mathbf{i}\in K^k} \xi(\mathbf{i}, s) \le \left(\sum_{\mathbf{i}\in K^1} \xi(\mathbf{i}, s)\right)^k$ which implies $\tau_\xi(s)$ is bounded since K^1 is finite.

By the third property of ξ,

$$c_-^{kh} \le \frac{\sum_{\mathbf{i}\in K^k} \xi(\mathbf{i}, s+h)}{\sum_{\mathbf{i}\in K^k} \xi(\mathbf{i}, s)} \le c_+^{kh}$$

Applying the definition of τ_ξ we get $c_-^h \le \tau(s+h)/\tau_\xi(s) \le c_+^h$ for $s \ge 0$ and $h > 0$. So $\tau_\xi(s)c_-^h \le \tau_\xi(s+h) \le c_+^h\tau_\xi(s)$ and τ_ξ is decreasing and continuous. Now, by the fourth property of τ_ξ, $\tau_\xi(0) = \lim_{k\to\infty} |K^k|^{1/k} \ge 1$. Finally if $s \ge -\log n/\log c_+$ then by submultiplicativity

$$\left(\sum_{\mathbf{i}\in K^k} \xi(\mathbf{i}, s) \right)^{1/k} \le \sum_{\mathbf{i}\in K^1} \xi(\mathbf{i}, s) \le nc_+^s \le 1 \Rightarrow \tau_\xi(s) \le 1$$

Hence there is a unique $s \ge 0$ such that $\tau(s) = 1$. □

The following measure on K is key, we will use it to define a measure on Λ and then show it satisfies the mass distribution principle.

$$M^s(A) = \lim_{k\to\infty} M_k^s(A) \text{ where } M_k^s(A) = \inf\left\{ \sum_{\mathbf{i}} \xi(\mathbf{i}, s) \ : \ A \subset \bigcup_{\mathbf{i}} I_{\mathbf{i}} \ , \ k \le |\mathbf{i}| < \infty \right\}$$

where $I_{\mathbf{i}}$ is the cylinder set of all infinite sequences which agree with $\mathbf{i} = (i_0, \ldots, i_n)$ on their first n terms. This is exactly a Method II construction as in lemma 1.4. Moreover, $\inf\{s : M^s(A) = 0\} = \sup\{s : M^s(A) = \infty\}$ since if $M^s(A) < \infty$, then for any $h > 0$ and any cover $\mathcal{Q}$ with $|\mathbf{i}| \le k$ for all $\mathbf{i} \in \mathcal{Q}$.

$$\sum_{\mathbf{i}\in\mathcal{Q}} \xi(\mathbf{i}, s+h)) \le c_+^k h \sum_{\mathbf{i}\in\mathcal{Q}} \xi(\mathbf{i}, s)$$

Giving $\mathcal{M}^t(A) = 0$, since $c_+^k h \to 0$ as $k \to \infty$.
The following lemma is often called *Frostman's Lemma*, a proof of which can be found in Rogers[8] Theorem 54. The proof is rather involved, so I will not give it here.

Lemma 2.13. *Let $A \subset \mathcal{A}^{\mathbb{N}}$ be a Borel subset and suppose $0 < M^s(A) \le \infty$ for some $s \ge 0$. Then there exists a compact set $A_0 \subseteq A$ and a constant $b > 0$ such that $0 < M^s(A_0) < \infty$ and $M^s(A_0 \cap [i_0, \ldots, i_n]) \le bc_{\mathbf{i}}^s$ for all $\mathbf{i} \in \mathcal{A}^{\mathbb{N}}$.*

Lemma 2.14. *The following numbers exist and are equal:*

1. The unique $s \ge 0$ such that $\tau(s) = 1$

2. $\inf\{s \geq 0 \,:\, \mathcal{M}^s(K) = 0\} = \sup\{s \geq 0 \,:\, \mathcal{M}^s(K) = \infty\}$
3. $\inf\{s \geq 0 \,:\, \sum_{k=1}^{\infty} \sum_{\mathbf{i} \in K^k} \xi(\mathbf{i}, s) < \infty\} = \sup\{s \geq 0 \,:\, \sum_{k=1}^{\infty} \sum_{\mathbf{i} \in K^k} \xi(\mathbf{i}, s) = \infty\}$

Moreover for this s, $M^s(K) \geq 1$

Proof. Equality in 2 was discussed above, and in 3 equality is clear.
1. $=$ 3. Follows since $\sum_{k=1}^{\infty} \sum_{\mathbf{i} \in K^k} \xi(\mathbf{i}, s)$ converges if $\tau(s) < 1$ and diverges if $\tau(s) > 1$.
2. $\leq$ 3. Taking the open cover K^k of K, $\sum_{k=1}^{\infty} \sum_{\mathbf{i} \in K^k} \xi(\mathbf{i}, s) < \infty$ implies $\mathcal{M}^s(K) = 0$ as

$$M_k^s(K) \leq \sum_{\mathbf{i} \in K^k} \xi(\mathbf{i}, s) \leq \sum_{j=k}^{\infty} \sum_{\mathbf{i} \in K^j} \xi(\mathbf{i}, s) \to 0 \text{ as } k \to \infty$$

1. $\leq$ 2. *and* $M^s(K) > 1$: Suppose $M^s(K) < 1$ for some $s > 0$, then there is a cover U of K by cylinders such that $\sum_{\mathbf{i} \in \mathcal{U}} \xi(\mathbf{i}, s) < 1$. By compactness, we assume $\mathcal{U}$ to be finite, and notice that for any t such that $0 < t < s$ we still have

$$\sum_{\mathbf{i} \in \mathcal{U}} \xi(\mathbf{i}, t) \leq 1 \tag{2.2}$$

Now take $p = \max\{|\mathbf{i}| \,:\, \mathbf{i} \in \mathcal{U}\}$ and define

$$\mathcal{U}_k = \{\mathbf{i}_1\mathbf{i}_2 \ldots \mathbf{i}_p \,:\, \mathbf{i}_j \in \mathcal{U}\, \forall 1 \leq j \leq p \,,\, |\mathbf{i}_1\mathbf{i}_2 \ldots \mathbf{i}_p| \geq k \,,\, |\mathbf{i}_1\mathbf{i}_2 \ldots \mathbf{i}_{p-1}| < k\}$$

Now,

$$\sum_{\mathbf{i} \in \mathcal{U}} \xi((i_0, \ldots, i_q)\mathbf{i}, s) \leq \xi((i_0, \ldots, i_q)) \sum_{\mathbf{i} \in \mathcal{U}} \xi(\mathbf{i}, s) \leq \xi((i_0, \ldots, i_q), s)$$

So by induction, $\sum_{\mathbf{i} \in \mathcal{U}_k} \xi(\mathbf{i}, s) \leq 1$. Now, if $\mathbf{i} \in K^{k+p}$, then $\mathbf{i} = \mathbf{i}'\mathbf{j}$ for some $\mathbf{i} \in \mathcal{U}_d$ and $|\mathbf{j}| \leq p$. Moreover, for any such $\mathbf{i}'$ there are at most m^p choices for such a $\mathbf{j}$ (where m is the number of functions in the iterated function system. Since $\xi(\mathbf{i}, s) \leq \xi(\mathbf{i}, s)$

$$\sum_{\mathbf{i} \in K^{k+p}} \xi(\mathbf{i}, s) \leq m^p \sum_{\mathbf{i}' \in \mathcal{U}_r} \xi(\mathbf{i}', s) \leq m^p$$

Since this is true for all k, $\lim_{r \to \infty} \left(\sum_{\mathbf{i} \in K^r} \xi(\mathbf{i}', s)\right)^{1/k} \leq 1$. □

For the rest of this section, $\xi(\mathbf{i}, s) = c_{\mathbf{i}}^s$ and we will write τ for τ_ξ. Notice that $\mathbf{i}, s \mapsto c_{\mathbf{i}}^s$ easily satisfies the conditions we imposed on ξ:

1. for all $s \geq 0$

$$\sum_{\mathbf{i} \in K^{k+l}} c_{\mathbf{i}}^s \leq \sum_{\mathbf{i} \in K^k \,,\, \mathbf{j} \in K^l} c_{\mathbf{ij}}^s = \left(\sum_{\mathbf{i} \in K^k} c_{\mathbf{i}}^s\right) \left(\sum_{\mathbf{i} \in K^l} c_{\mathbf{i}}^s\right)$$

   Hence $\sum_{\mathbf{i} \in K^k} c_{\mathbf{i}}^s$ is a submultiplicative sequence.
2. is obvious.
3. Let $c_- = \min_i c_i$ and $c_+ = \max_i c_i$.
4. is obvious.

We now move on to prove the main theorem of this section, as with most of these techniques, the upper bound is easy to estimate.

Lemma 2.15.

$$\dim_{\mathcal{H}} \Lambda \leq \underline{\dim}_B \Lambda \leq \overline{\dim}_B \Lambda \leq s$$

Moreover if $M^s(K) < \infty$ *then* $\mathcal{H}^s(\Lambda) < \infty$

Proof. If $\mathcal{Q}$ is such that $K \subseteq \cup_{\mathbf{i} \in \mathcal{Q}} I_{\mathbf{i}}$ where $|\mathbf{i}| \geq k$ for all $\mathbf{i} \in \mathcal{Q}$, then $\Lambda \subseteq \cup_{\mathbf{i} \in \mathcal{Q}} S(\mathbf{i})$ so for all $\delta \leq c_+^k$.

$$\mathcal{H}_\delta^s(\Lambda) \leq \sum_{\mathbf{i} \in \mathcal{Q}} |S(\mathbf{i})|^s = |B|^s \sum_{\mathbf{i} \in \mathcal{Q}} c_{\mathbf{i}}^s \leq |B|^s \mathcal{M}_k^s(E)$$

Now if $k \to \infty$, then $\delta \to 0$ and we get $\mathcal{H}^s(\Lambda) \leq |B|^s \mathcal{M}^s(K)$. □

Theorem 2.16. *If* Λ *is sub-self-similar with respect to* $\{S_1, \ldots, S_m\}$*, similarities satisfying the open set condition, and with contraction ratios* $\{c_0, \ldots, c_m\}$*, and* s *the unique number such that* $\tau(s) = 1$*, then* $\dim_{\mathcal{H}} \Lambda = \underline{\dim}_B \Lambda = \overline{\dim}_B \Lambda = s$*.*

We will only prove this for the Hausdorff dimension, and not the box-counting dimension.

Proof. Since we already have the upper bound from Lemma 2.15, it suffices to show $\mathcal{H}^s(\Lambda) > 0$. By Lemma 2.14 we know that $M^s(K) > 0$ so by Lemma 2.13 there is a compact subset A of K such that the measure $\mu(U) = \mathcal{M}^s(A \cap U)$ for $U \subset \mathcal{A}^{\mathbb{N}}$ is supported by K and satisfies $\mu(K) > 0$ and

$$\mu(I_{\mathbf{i}}) \leq bc_{\mathbf{i}}^s \text{ for all } \mathbf{i} \in K \tag{2.3}$$

Now, set $\nu(U) = \mu\{\mathbf{i} \,:\, S(\mathbf{i}) \in U\}$ which is a measure by Lemma 1.30. Notice ν is supported by $S(K) = \Lambda$. Now, let V be a feasible open set (as in Definition 2.1), and let U be a subset satisfying $0 < |U| \leq |V|$. Finally, let $\mathcal{Q}$ be the set

$$\mathcal{Q} = \{(i_0, \ldots, i_k) \,:\, c_{i_0}c_{i_1}\cdots c_{i_k}|V| < |U| \,,\, c_{i_0}c_{i_1}\cdots c_{i_{k-1}}\}$$

Now, since $c_-|U| \leq |S_{\mathbf{i}}(V)| < |U|$ for $\mathbf{i} \in \mathcal{Q}$, there are finitely many indices in

$$\mathcal{Q}_0 = \{\mathbf{i} \in \mathcal{Q} \,:\, U \cap S_{\mathbf{i}}(\overline{V}) \neq \emptyset\}$$

Write $q_0 = |\mathcal{Q}_0$, and notice q_0 is independent of U by Lemma 2.2. Notice also that $\{S_{\mathbf{i}}(V) \,:\, \mathbf{i} \in \mathcal{Q}\}$ is a pairwise disjoint collection of sets (by OSC).
If $S(\mathbf{j}) \in U$ then $\mathbf{j}|_k \in \mathcal{Q}$ for some k so $\mathbf{j} \in I_{\mathbf{i}}$ for some $\mathbf{i} \in \mathcal{Q}_0$ and then by 2.3

$$\nu(U) = \sum_{\mathbf{i} \in \mathcal{Q}_0} \mu\{\mathbf{j} \in I_{\mathbf{i}}\} \leq b \sum_{\mathbf{i} \in \mathcal{Q}_0} c_{\mathbf{i}}^s \leq bb_1|V|^{-s}|U|^s$$

So since ν is supported by Λ, we are done by the mass distribution principle (Lemma 1.16)

□

2.3 Subshifts of Finite Type

The next section provides what is sometimes an easier method of calculating the Hausdorff dimension than that of theorem 2.16 in the case that the subshift used to model the IFS is a subshift of finite type. This technique is due to L. Block and J. Keesling [7]

Example 2.17. *Recall the Modified Sierpiński Triangle of Example 2.19, defined as the sub-self-similar set Λ with respect to the similarities $\{S_1, S_2, S_3\}$, where*

$$S_1 = T, S_2 = T + \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, S_3 = T + \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ where } T = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

Where we restrict the order in which we can apply the similarities by not allowing S_1 to be composed with itself, ie. we take the transition matrix

$$B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

and then $\Lambda = \cup_{\mathbf{i} \in \mathcal{A}_B^{\mathbb{N}}} S(\mathbf{i})$. Using theorem below, we consider the matrix

$$M(s) = \begin{pmatrix} 0 & c_1^s & c_1^s \\ c_2^s & c_2^s & c_2^s \\ c_3^s & c_3^s & c_3^s \end{pmatrix} = \frac{1}{2^s}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \frac{1}{2^s}B$$

B has largest real eigenvalue $\lambda = 1+\sqrt{3}$ so by the theorem below the Hausdorff dimension of our modified triangle is given by $\log 1+\sqrt{3}/\log 2 \approx 1.45$. (Compare this with the Hausdorff dimension of the regular Sierpiński triangle $\log 3/\log 2 \approx 1.58$)

Let $S_1, \ldots, S_m$ be a set of contractions with contraction ratios $\{c_1, \ldots, c_m\}$ and Λ a sub-self-similar set with respect to $S_1, \ldots, S_m$. Then let $K \subseteq \mathcal{A}^{\mathbb{N}}$ be the subshift given by lemma 2.9, such that $K \subseteq \sigma(K)$ and $\Lambda = S(K)$. Let ρ be the metric on $\mathcal{A}$ associated to $\{c_1, \ldots c_m\}$

Theorem 2.18. *If K is a subshift of finite type given by $K = \mathcal{A}_B$ for some transition matrix $B = (b_{ij})$ then $\dim_{\mathcal{H}}(\Lambda) = \beta$, where β is the unique value of s for the which the matrix $M(s) = (b_{ij}c_j^s)_{ij}$ has largest real eigenvalue $\lambda = 1$.*

Proof. We prove this in the case where all the c_i satisfy $c_i < \frac{1}{2}$, the general case can be proved by reducing to this case, via redefining the metric ρ on $\mathcal{A}^{\mathbb{N}}$. We also will only prove it in the case where the graph associated to the matrix A is strongly connected (there is a path from every vertex to every other vertex), as this simplifies the proof greatly. A full proof can be found in Block&Keesling [7]. Now, notice that $B^k(s)_{x,y}$ is the number of 'length-k paths' from x to y on the directed graph determined by B, and that

$$M^k(s)_{i,j} = \sum_{\mathbf{i}\in K^k} c_{i_1}\cdots c_{i_k} \text{ with } i_0 = x \ , \ i_k = y$$

This then gives that

$$(c_1^s c_2^s \cdots c_n^s)\, M(s)^{k-1} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \sum_{\mathbf{i}\in K^k} c_{\mathbf{i}}^k$$

So

$$\tau(s) = \lim_{k\to\infty} \left(\left(c_1^2 c_2^2 \cdots c_n^s\right) M(s)^{k-1} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \right)^{\frac{1}{k}}$$

Now since the graph associated to A is strongly connected, $M(s)^k$ is a non-negative irreducible real matrix, and by the Perron-Frobenius Theorem it has a largest real eigenvalue λ with $\lim_{k\to\infty} M(s)^k/\lambda^k = vw^T$, where all elements of v and w are non-negative. So

$$\lim_{k\to\infty} \left(\lambda^{-k} \left(c_1^2 c_2^2 \cdots c_n^s\right) M(s)^k \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \right) = a_0 \ \Rightarrow \ \lim_{k\to\infty} \left(\left(c_1^2 c_2^2 \cdots c_n^s\right) M(s)^k \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \right) = \lim_{k\to\infty} a_0\lambda^k$$

For a_0 some positive constant. Hence $\tau(s) = 1$ if and only if the largest eigenvalue of $M(s)$ is 1. By Theorem 2.16 of the previous section, we know the Hausdorff dimension of Λ is the unique value of s such that $\tau(s) = 1$, which completes the proof. $\square$

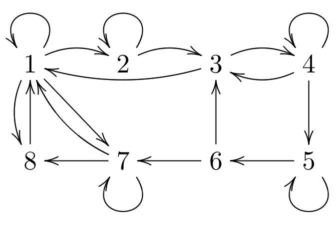
Example 2.19. *Let $S_i = T_i + a_i$ be contracting similarities given by the following.*

$$T_i = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \textit{ for all } i$$

$$a_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ a_2 = \begin{pmatrix} \frac{1}{3} \\ 0 \end{pmatrix}, \ a_3 = \begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix}, \ a_4 = \begin{pmatrix} 0 \\ \frac{1}{3} \end{pmatrix}$$

$$a_5 = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}, \ a_6 = \begin{pmatrix} 0 \\ \frac{2}{3} \end{pmatrix}, \ a_7 = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}, \ a_8 = \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$$

Notice that if we allowed all combinations, this would give a regular Sierpiński carpet. Now let $\mathcal{A}_B$ be the subshift of finite type generated by the following directed graph, and denote the transition matrix of this graph B.



The matrix $M(s) = (b_{i,j}c_j^s) = \frac{1}{3^s}B$, and B has largest real eigenvalue $\lambda \approx 2.4652$, so by theorem 2.18 it has Hausdorff dimension $\dim_{\mathcal{H}} \Lambda \approx 0.8213$.

3 Self-Affine Sets

In this section we remove the open set condition and restrict to the case where the S_i are affine transformations, ie. $S_i(x) = T_i(x) + a_i$ where $T_i : \mathbb{R}^n \to \mathbb{R}^n$ are linear maps and $a_i \in \mathbb{R}^n$. We will usually write $a = (a_1, \ldots, a_m) \in \mathbb{R}^{nm}$ when we want to refer to all the translations at once. We can still work with subshifts $K \subseteq \mathcal{A}^{\mathbb{N}}$ which correspond to sub-self-similar sets via Lemma 2.9. A self-similar set generated by an iterated function system of affine transformations is called a *self-affine set.*

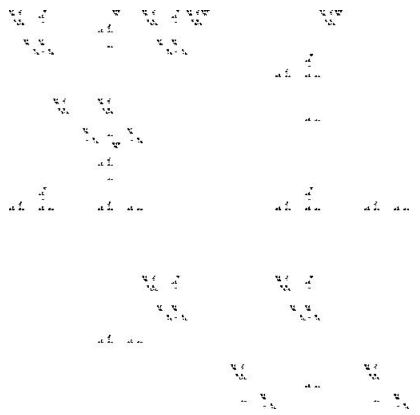


Figure 7: Modified Sierpiński Carpet from transition matrix B: $\dim_{\mathcal{H}}(\Lambda) \approx 0.8213$

From here on we fix a subshift K, fix the T_i and write $\Lambda(a)$ for the invariant set corresponding to $T_i = S_i + a_i$. We will also write S_a for the map S as in previous sections corresponding to a. Ideally we want an explicit formula dependant on the T_i and a_i, but a general solution of this kind does not currently exist. There is however a result by Falconer which gives the Hausdorff dimension in terms of the T_i for almost all a.

Example 3.1. *Firstly an example of just how badly behaved self-affine sets can be, consider the transformations*

$$T_1 = T_2 = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \quad a_1 = \begin{pmatrix} 0 \\ \frac{1}{4} \end{pmatrix} \quad a_2 = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{4} \end{pmatrix}$$

This gives $\Lambda = \mathcal{C} \times \{\frac{1}{2}\} \subset [0,1]^2$*, ie. a cantor set, so we know by previous calculations that*

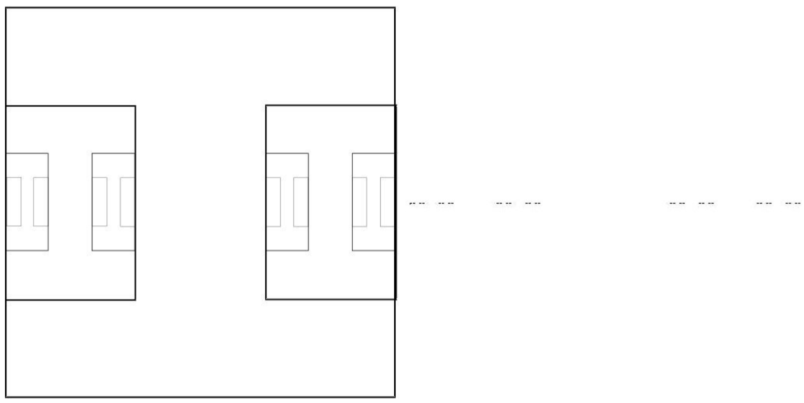


Figure 8: $\dim_{\mathcal{H}}(\Lambda) = \log 2/\log 3$

$\dim_{\mathcal{H}}(\Lambda) = \log 2/\log 3 \approx 0.63$.
However, if we adjust the values of a_i in the affine transformations to

$$a_1 = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} \quad a_2 = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{4} \end{pmatrix}$$

Then the Hausdorff dimension of the new limit set $\tilde{\Lambda}$ as $\dim_{\mathcal{H}}(\tilde{\Lambda}) \geq \dim_{\mathcal{H}}(\mathrm{proj}(\tilde{\Lambda})) = 1$*, where* proj *is projection onto the second coordinate (by lemma 1.11, since the projection map is Lipschitz). Notice that although both transformations have the same T_i the change in a_i causes a substantial change in the Hausdorff Dimension. In fact the situation is worse than this, the main theorem of this section will show that* $\dim_{\mathcal{H}} \Lambda(a)$ *is a constant for almost all a with respect to the Lebesgue measure on* $\mathbb{R}^{nk}$*. Combining the above example with that theorem shows* $\dim_{\mathcal{H}}(\Lambda(a))$ *is not continuous in a.*

3.1 The Singular Value Function

Firstly we just need a quick aside on the *Singular Value Decomposition* of matrices (a discussion of this can be found in Strang&Gilbert[13]). If a matrix T is real valued and non-singular, T admits a factorisation $T = P_1 \Sigma P_2^*$, where P_1 and P_2 are orthogonal real-valued matrices and Σ is diagonal real-valued matrix. Now,

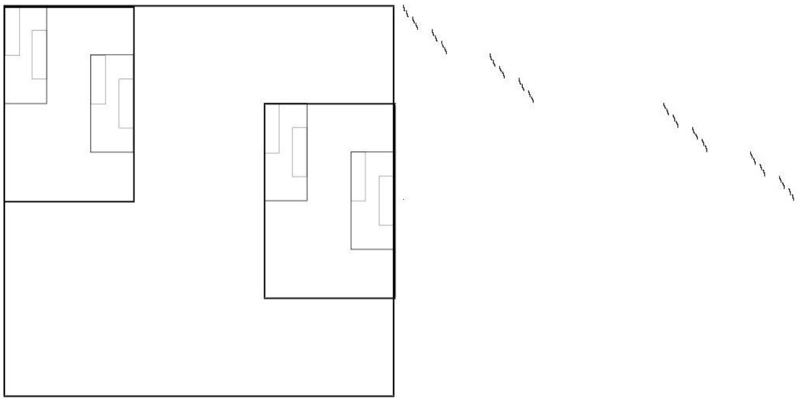


Figure 9: $\dim_{\mathcal{H}}(\tilde{\Lambda}) \geq 1$

$T^*T = P_2\Sigma^*P_1^*P_1\Sigma P_2^* = P_2\Sigma^*\Sigma P_2^*$, so since $\Sigma^*\Sigma$ is a diagonal matrix with $\alpha_1^2, \ldots, \alpha_n^2$ on the diagonal, it is clear that the α_i are the square roots of the non-zero eigenvalues of TT^* and T^*T, and the columns of P_1 are an orthogonal eigenbasis. Writing $\mathcal{P}_1$ for the columns of P_1 and $\mathcal{P}_2$ for the columns of P_2, notice that P_1 and P_2 form two orthogonal bases with the property that the map T takes the ith element of $\mathcal{P}_1$ to a non-zero multiple of the ith element of $\mathcal{P}_2$.
The diagonal elements of Σ are known as the *Singular Values* of T, and correspond to the lengths of the semi-axis of the ellipsoid $T(B)$ where B is the unit ball. For example the singular value decomposition of $T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ is

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = P_1\Sigma P_2 \approx \begin{pmatrix} -0.52573 & -0.85065 \\ -0.85065 & 0.52573 \end{pmatrix} \begin{pmatrix} 1.61803 & 0 \\ 0 & 0.61803 \end{pmatrix} \begin{pmatrix} -0.85065 & -0.52573 \\ -0.52573 & 0.85065 \end{pmatrix}$$

So the singular values of T are $\{0.61803, 1.61803\}$. Notice that P_1 is a reflection in the y-axis, and a rotation of 1.0172 about the origin, P_2 is a reflection in the y-axis and a rotation of 0.5536 about the origin.
Much of the proof revolves around studying what Falconer calls the singular value function, defined below.

Definition 3.2. *Let $\{\alpha_i\}_{i=0}^n$ be the lengths of the semi-axis of $T(B)$, where B is the unit ball in $\mathbb{R}^n$. Order the α_i such that $0 \leq \alpha_n \leq \ldots \alpha_1 < 1$. The* singular value function *is defined as*

$$\phi^s(T) = \begin{cases} \alpha_1\alpha_2\ldots\alpha_{m-1}\alpha_m^{s-m+1} & \text{where } s \leq n \\ (\alpha_1\alpha_2\ldots\alpha_n)^{s/n} = (\det T)^{s/n} & s > n \end{cases}$$

Where $m-1 < s \leq m, m \in \mathbb{N}$

Lemma 3.3. 1. *ϕ^s is continuous and strictly decreasing in s.*
2. *If $s \in \mathbb{N}$, with $0 \leq s \leq n$, then $\phi^s(T) = \alpha_1 \cdots \alpha_s = \sup \mathcal{L}^s(T(E))/\mathcal{L}^s(E)$, where the supremum is over the set of s-dim ellipsoids in $\mathbb{R}^n$ and $\mathcal{L}^s$ denotes the s-dimensional Lebesgue measure.*
3. *ϕ^s is submultiplicative, ie. $\phi^s(TU) \leq \phi^s(T)\phi^s(U)$ for all $T, U \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$.*

Note than when we speak of the s-dimensional ellipsoid in $\mathbb{R}^n$, we mean an ellipse in some s-dimensional hyperplane of $\mathbb{R}^n$, and identifying that hyperplane with $\mathbb{R}^s$ gives us the s-dimensional Lebesgue measure of that ellipsoid.

Proof. 1. Since the transformations P_1 and P_2 are orthogonal, they preserve area and hence it is sufficient to consider the effect of Σ on an arbitrary s-dimensional ellipsoid. Now, if E is the unit ball in $\mathbb{R}^s \subset \mathbb{R}^n$ then $\mathcal{L}^s(\Sigma(E)) = \alpha_1, \ldots, \alpha_s$. (We are assuming that $\alpha_1, \ldots, \alpha_s$ are the first s entries in the diagonal matrix Σ). If E is an arbitrary s-dimensional ellipsoid in some other hyperplane in $\mathbb{R}^n$ then it is clear that T will shrink E by a greater amount since the singular values smaller than α_s will have an effect.

2. If E is an s-dimensional ellipsoid, $s \in \mathbb{N}$ and $1 \leq s \leq n$, then

$$\mathcal{L}^s(TU(E)) \leq \phi^s(T)\mathcal{L}^s(U(E)) \leq \phi^s(T)\phi^s(U)\mathcal{L}^s(E) \tag{3.1}$$

Applying this to 2. gives the result. Now, if $s \in \mathbb{R}$ and $0 < s < n$ then let m be the integer such that $m-1 < s \leq m$ so

$$\phi^s(T) = \alpha_1 \cdots \alpha_{m-1}\alpha_m^{s-m+1} = (\alpha_1 \cdots \alpha_m)^{s-m+1}(\alpha_1 \cdots \alpha_{m-1})^{m-s} = (\phi^m(T))^{s-m+1}\left(\phi^{m-1}(T)\right)^{m-s}$$

So applying the above to (3.1) gives the result.
Finally if $s > n$, then $\phi^s(TU) = \det(TU)^{s/n} = \det(T)^{s/n}\det(U)^{s/n} = \phi^s(T)\phi^s(U)$.

□

Lemma 3.4. *Let $s \notin \mathbb{N}$, $0 < s < n$. Then there exists a number $c < \infty$ dependant on n,s and r such that*

$$\int_{B_r} \frac{dx}{\|Tx\|^s} \leq \frac{c}{\phi^s(T)}$$

for all non-singular $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, where $\mathcal{L}$ denotes linear mappings and B_r is the ball of radius r in $\mathbb{R}^n$

Proof.

$$I = \int_{B_r} \frac{dx}{< Tx, Tx >^{s/2}} = \int_{B_r} \frac{dx}{< x, T^*Tx >^{s/2}} = \int_{B_r} \frac{d(x_1 \cdots x_n)}{< (x_1, \dots, x_n), (\alpha_1^2 x_1, \dots, \alpha_n^2 x_n) >^{s/2}}$$

$$= \int \cdots \int_{B_r} \frac{dx_1 \cdots dx_n}{(\alpha_1^2 x_1^2 + \cdots + \alpha_n^2 x_n^2)^{s/2}}$$

where T^* is the adjoint operator of T, and $x_1, \dots, x_n$ are coordinates in the direction of the eigenvectors of T*T (see the remark at the beginning of the section). Substituting $y_i = \alpha_i x_i / r$ to give

$$I \leq \int \cdots \int_{B_r} \frac{dx_1 \cdots dx_n}{r^s (y_1^2 + \cdots y_n^2)^{s/2}} \leq \int \cdots \int_P \frac{r^{n-s} \alpha_1^{-1} \cdots \alpha_n^{-1} dx_1 \cdots dx_n}{(y_1^2 + \cdots + y_n^2)^{s/2}}$$

where $P = \{y = (y_1, \dots, y_n) : |y_i| \leq \alpha_i\}$. Next let $m \in \mathbb{N}$ be such that $m - 1 < s \leq m$ and write

$$P_1 = \{y \in P \ : \ y_1^2 + \cdots + y_m^2 \leq 4\alpha_m^2\} \ , \ P_2 = \{y \in P \ : \ y_1^2 + \ldots + y_{m-1}^2 > \alpha_m^2\}$$

so $P \subset P_1 \cup P_2$ since $|y_m| \leq \alpha_m$ in P. Hence

$$r^{s-n} \alpha_1 \cdots \alpha_n I \leq \int \cdots \int_{P_1} \frac{dy_1 \cdots dy_n}{(y_1^2 + \cdots y_m^2)^{s/2}} + \int \cdots \int_{P_2} \frac{dy_1 \cdots dy_n}{(y_1^2 + \cdots y_m^2)^{s/2}}$$

By transforming the first m, and m-1 coordinates respectively into polar coordinates we get

$$r^{s-n} \alpha_1 \cdots \alpha_n I \leq \int_0^{2\pi} \cdots \int_0^{2\pi} \int_0^{2\alpha_m} \int \cdots \int \mathcal{D}_1 r^{-s} d\theta_1 \cdots d\theta_{m-1} \, dr \, dy_{m+1} \cdots dy_n$$

$$+ \int_0^{2\pi} \cdots \int_0^{2\pi} \int_{a_m}^{\infty} \int \cdots \int \mathcal{D}_2 r^{-s} d\theta_1 \cdots d\theta_{m-2} \, dr \, dy_m \cdots dy_n$$

where $\mathcal{D}_1 = r^{m-1} \sin^{m-2}\theta_1 \sin^{m-3}\theta_2 \dots \sin\theta_{m-2}$ and $\mathcal{D}_2 = r^{m-2} \sin^{m-3}\theta_1 \dots \sin\theta_{m-3}$

$$r^{s-n} \alpha_1 \cdots \alpha_n I \leq c_1 \alpha_{m+1} \cdots \alpha_n \int_0^{2\alpha_m} r^{-s} r^{m-1} dr + c_2 \alpha_m \cdots \alpha_n \int_{a_m}^{\infty} r^{-s} r^{m-2} dr$$

$$\leq c_1' \alpha_{m+1} \cdots \alpha_n \alpha_m^{m-s} + c_2' \alpha_m \cdots \alpha_n \alpha_m^{m-s-1}$$

where c_1, c_2, c_1', c_2' are constants independent of α. So

$$I r^{s-n} \leq c_1' (\alpha_1 \dots \alpha_{m-1} \alpha_m^{s-m+1})^{-1} + c_2' (\alpha_1 \dots \alpha_{m-1} \alpha_m^{s-m+1})^{-1} = \frac{(c_1' + c_2')}{\phi^s(T)}$$

□

The next lemma was originally proved by Falconer for the case $\|T_i\| < \frac{1}{3}$, but then improved by Solomyak[12] to the case $\|T_i\| < \frac{1}{2}$. A discussion of this condition follows in section 3.3

Lemma 3.5. *If $s \notin \mathbb{N}$, $0 < s < n$ and $\eta < \frac{1}{2}$ where $\eta = \max_i \|T_i\|$, then there exists some $c < \infty$, such that for all $\mathbf{i} \neq \mathbf{j} \in \mathcal{A}^{\mathbb{N}}$*

$$\int_{a \in B_r \subset \mathbb{R}^{nk}} \frac{da}{|S_a(\mathbf{i}) - S_a(\mathbf{j})|^s} \leq \frac{c}{\phi^s(T_{\mathbf{p}})}$$

where $\mathbf{p} = \mathbf{i} \wedge \mathbf{j}$ denotes the longest subsequence which agrees with $\mathbf{i}$ and $\mathbf{j}$.

Proof. Write $\mathbf{i} = \mathbf{p}, \mathbf{i}'$, $\mathbf{j} = \mathbf{p}, \mathbf{j}'$ and $p = |\mathbf{p}|$. Suppose (without loss of generality) also that the first terms of $\mathbf{i}'$ and $\mathbf{j}'$ are 1 and 2 respectively. Now

$$S_a(\mathbf{i}') - S_a(\mathbf{j}') = a_1 - a_2 + \left(T_{\mathbf{i}_1'} a_{\mathbf{i}_2'} + T_{\mathbf{i}_1'} T_{\mathbf{i}_2'} a_{\mathbf{i}_3'} + \cdots\right) - \left(T_{\mathbf{j}_1'} a_{\mathbf{j}_2'} + T_{\mathbf{j}_1'} T_{\mathbf{j}_2'} a_{\mathbf{j}_3'} + \cdots\right) = a_1 - a_2 + E_1(a_1) + \dots E_m(a_m)$$

where the E_i are linear transformations $\mathbb{R}^n \to \mathbb{R}^n$. Now, we can choose $\alpha = 1$ or 2 such that for some $2 \le n \le \infty$, $\mathbf{i}'_k$ and $\mathbf{j}'_k$ are not both equal to α for all $k < n$ and (if $\alpha < \infty$) $\mathbf{i}'_n \neq \alpha \neq \mathbf{j}'_n$. Hence

$$|E_\alpha| \le \sum_{k=2}^{n-1} \eta^{k-1} + \sum_{k=n+1}^{\infty} 2\eta^{k-1} \le \frac{\eta}{1-\eta} < 1$$

Since $\eta < \frac{1}{2}$. Also notice that by the standard result on linear operators (see Eidelman, Milman and Tsolomitis [16] Section 4.7) $\| \pm 1 + E_\alpha\| < 1$ is invertible and $\|(\pm 1 + E_\alpha)^{-1}\| < (1-\eta)/(1-2\eta)$. Without loss of generality, assume $\|1 + E_\alpha\|$ is invertible, and take the coordinate transformation (where $\beta = 1 - \alpha$)

$$y = (a_\alpha + E_\alpha(a_\alpha)) - a_\beta + E_\beta(a_\beta) - a_3 + E_3(a_3) - \dots\ ,\ a_\beta = a_\beta\ ,\ a_3 = a_3\ \cdots\ a_k = a_k$$

Noticing that

$$\Big(\|(a_\alpha + E_\alpha(a_\alpha)) - a_\beta + E_\beta(a_\beta) - a_3 + E_3(a_3) - \dots\| < (2+k)r \text{ and } a_i \in B_r^n\ \forall i \neq \alpha\Big) \Rightarrow (a_1, \dots a_k) \in B_r^{nk}$$

We can obtain

$$\int_{a\in B_r^{nk}} \frac{da}{|S_a(\mathbf{i}) - S_a(\mathbf{j})|^s} = \int_{a\in B_r^{nk}} \frac{da}{T_{\mathbf{p}}(S_a(\mathbf{i}') - S_a(\mathbf{j}'))^s} \le \int_{\substack{y\in B_{(2+k)r}\\ a_i \in B_r^n}} \frac{dy\, da_\beta\, da_3 \cdots da_k}{|T_{\mathbf{p}}(y)|^s} \le \frac{c}{\phi^s(T_{\mathbf{p}})}$$

Where the final inequality follows from Lemma 3.4. □

3.2 Falconer's Theorem

We define a measure on the sequence space using a very similar method to in the sub-self-similar section.

$$\mathcal{M}_k^s(A) = \inf\left\{\sum_{\mathbf{i}} \phi^s(T_{\mathbf{i}})\ :\ A \subseteq \bigcup_{\mathbf{i}} I_{\mathbf{i}}\ ,\ |\mathbf{i}| \ge k\right\} \text{ and } M^s(A) = \lim_{k\to\infty} M_k^s(A)$$

Which is a countably-subadditive measure by Method II (Lemma 1.4). In fact, this is the construction in section 2.2, with $\xi(\mathbf{i}, s) = \phi^s(T_{\mathbf{i}})$. Indeed this choice of ξ satisfies our four conditions

1. By lemma 3.3
2. By lemma 3.3
3. Let $c_- = \alpha_n$ and $c_+ = \alpha_1$
4. Is clear since $\phi^s(T) = 0$ for all T.

We can then conclude the following version of Lemma 2.14

Lemma 3.6. *The following numbers all exist and are equal:*

1. $\inf\{s\ :\ \mathcal{M}^s(K) = 0\} = \sup\{s\ :\ \mathcal{M}^s(K) = \infty\}$
2. *The unique $s > 0$ such that* $\tau(s) = \lim_{k\to\infty}\left(\sum_{\mathbf{i}\in K^k} \phi^s(T_{\mathbf{i}})\right)^{1/k} = 1.$
3. $\inf\left\{s\ :\ \sum_{\mathbf{i}\in K^F} \phi^s(T_{\mathbf{i}}) < \infty\right\} = \sup\left\{s\ :\ \sum_{\mathbf{i}\in K^F} \phi^s(T_{\mathbf{i}}) = \infty\right\}$

We now proceed to prove the main theorem of this section, this upper bound is proved in a similar way as Theorem 2.4.

Lemma 3.7. *If $\mathcal{M}^t(K) < \infty$ then $\mathcal{H}^t(\Lambda(a)) < \infty$ and in particular $\dim_{\mathcal{H}} \Lambda(a) \le s$ where s is the unique number such that $\tau(s) = 1$.*

Proof. Given some $\delta > 0$ there exists r such that $|S(\mathbf{i})| < \delta$ for all $|\mathbf{i}| > r$. Choose a covering set $\mathcal{U}$ of K such that $|\mathbf{i}| \ge r$ for each $\mathbf{i} \in \mathcal{U}$. So $\Lambda(a) \subseteq \cup_{\mathbf{i}\in\mathcal{U}} S(\mathbf{i})$.
For $\mathbf{i} \in K^F$, $S(\mathbf{i})$ is contained in a fixed parallelepiped P with sides of length $2|B|\alpha_1, \dots, 2|B|\alpha_n$. If m is the integer such that $m - 1 < s \le m$ we divide P into at most

$$\left(4|B|\frac{\alpha_1}{\alpha_m}\right)\cdots\left(4|B|\frac{\alpha_{m-1}}{\alpha_m}\right)(4|B|)^{n-m+1}$$

cubes of side length α_m, ie. diameter $\sqrt{n}\alpha_m$.

$$\inf\{\sum |U_i|^s\ :\ U_i \text{ a } \sqrt{n}\delta \text{ cover of } \Lambda(a)\} \le \sum_{\mathbf{i}\in\mathcal{U}} \{(4|B|)^n \alpha_1\alpha_2\dots\alpha_{m-1}\alpha_m^{1-m}(\sqrt{n}\alpha_m)^s$$

$$\le (4|B|\sqrt{n})^n \sum_{\mathbf{i}\in\mathcal{U}} \phi^s(T_{\mathbf{i}})$$

Taking the infimum over both all such sets $\mathcal{U}$ gives $\inf\{\sum |U_i|^s\ :\ U_i \text{ a } \sqrt{n}\delta \text{ cover of } \Lambda(a)\} \le (4|B|\sqrt{n})^n \mathcal{M}_r^s(K)$.
Now, letting $\delta \to 0$, so $r \to \infty$ gives $\mathcal{H}^s(\Lambda(a)) \le \mathcal{M}^s(K)$ by Lemma 3.6. □

Now, we start work on the lower bound, which will use thermodynamic formalism. As a preliminary, notice that $S_a(\mathbf{i})$ is continuous in a for fixed $\mathbf{i}$. This follows since we could have defined S_a by taking

$$S_a(\mathbf{i}) = \bigcap_{r=0}^{\infty} S_{i_0} \circ \cdots \circ S_{i_r}(z) = \lim_{r\to\infty} (T_{i_0} + a_0) \circ \cdots \circ (T_{i_r} + a_r)(0)$$

(In fact we could chose any vector including 0, but 0 suits the following calculation). So

$$S_a(\mathbf{i}) = \lim_{r\to\infty} a_0 + T_{i_0}(a_1) + T_{i_0} \circ T_{i_1}(a_2) + T_{i_0} \circ T_{i_1} \circ \cdots \circ T_{i_{r-1}}(a_r)$$

$$|S_a(\mathbf{i}) - S_b(\mathbf{j})| = \left|\lim_{r\to\infty} a_0 - b_0 + T_{i_0}(a_1) - T_{i_0}(b_1) + \ldots + T_{i_0} \circ T_{i_1} \circ \cdots \circ T_{i_{r-1}}(a_r) - T_{i_0} \circ T_{i_1} \circ \cdots \circ T_{i_{r-1}}(b_r)\right|$$

$$\leq \sum_{r=0}^{\infty} \alpha_1^r \max_i \{a_i - b_i\} = B \max_i \{a_i - b_i\}$$

For some constant B, since $|T_{i_0} \circ \cdots \circ T_{i_k}(a_{k+1}) - T_{i_0} \circ \cdots \circ T_{i_k}(b_{k+1})| \leq \alpha_1^k |a_{k+1} - b_{k-1}|$ and $\alpha_1 < 1$.

Lemma 3.8. *Suppose μ is a measure on the Borel σ-algebra $\mathcal{B}$ of $\mathcal{A}^{\mathbb{N}}$ with $0 < \mu(K) < \infty$ and for some $s < n$*

$$\int_K \int_K \int_{a\in B_r \subset \mathbb{R}^{nk}} \frac{da\, d\mu(\mathbf{i})\, d\mu(\mathbf{j})}{|S_a(\mathbf{i}) - S_a(\mathbf{j})|^2} < \infty$$

then for almost all $a \in B_r$ (in the Lebesgue measure sense) we have that $\dim \Lambda(a) \geq s$

Proof. Notice $(\mathbf{i}, \mathbf{j}, a) \mapsto \min\{r, |S_a(\mathbf{i}) - S_a(\mathbf{j})|^{-s}\}$ is a continuous function in $\mathbf{i}, \mathbf{j}$ and a for any fixed r. This function is measurable since $\mu \times \mu \times \mu_{\mathcal{L}}$ is a metric countably sub-additive measure, since it is a product of metric measures (where $\mu_{\mathcal{L}}$ is the Lebesgue measure). For such measures continuous real valued functions are measurable (see [20]Theorem 19.1). Now define

$$\psi(a, \mathbf{i}, \mathbf{j}) = \lim_{r\to\infty} \min\{r, |S(\mathbf{i}) - S(\mathbf{j})|^{-s}\}$$

ψ is Borel measurable on $\mathbb{R}^{nk} \times K \times K$ (by [20] Theorem 20.3), so by Fubini's theorem (see [20] Thm 29.7) for almost all $a \in B_r$

$$\int_K \int_K \frac{d\mu(\mathbf{i}) d\mu(\mathbf{j})}{|S_a(\mathbf{i}) - S_a(\mathbf{j})|^s} < \infty$$

For these a we define ν on $\mathbb{R}^n$ by $\nu(U) = \mu\{\mathbf{i} : S_a(\mathbf{i}) \in U\}$. This is a measure when restricted to the Borel σ-algebra, by lemma 1.30. Moreover, ν is supported by $\Lambda(a)$. So $\Lambda(a)$ supports a mass distribution satisfying the conditions of Thermodynamic Formalism (Lemma 1.20) and we may conclude $\Lambda(a) \geq s$. □

Theorem 3.9. *If $|T_i| < \frac{1}{3}$, for $1 \leq i \leq m$. Then for almost all $a \in \mathbb{R}^{nm}$,* $\dim_{\mathcal{H}} \Lambda(a) = \min\{n, d(T_1, \ldots, T_k)\}$, *where $d(T_1, \ldots, T_k) = \inf\{s : \sum_{i\in J} \phi^s(T_i) < \infty\}$*

Proof. If $t \notin \mathbb{N}$, $0 < t < \min\{n, d(T_1, \ldots, T_k)\}$ and choose s such that $t < s < \min\{n, d(T_1, \ldots, T_k)\}$. Now $\mathcal{M}^s(K) = \infty$ so by Lemma 2.13 there is a compact set $K \subset \mathcal{A}^{\mathbb{N}}$ such that $0 < \mathcal{M}^s(K) < \infty$ and for all $\mathbf{i} \in \mathcal{A}^F$.

$$\mathcal{M}^s(K \cap I_{\mathbf{i}}) \leq c_1 \phi^s(T_{\mathbf{i}}) \tag{3.2}$$

Now we define a measure μ on $\mathcal{A}^{\mathbb{N}}$ by $\mu(A) = \mathcal{M}^s(K \cap A)$ so $\mu(I_{\mathbf{i}}) \leq c_1 \phi^s(T_{\mathbf{i}})$ for all $\mathbf{i} \in \mathcal{A}^F$. Recall that since $\mathcal{M}$ was a measure constructed via Method II, Borel sets are $\mathcal{M}$-measurable, and hence Borel sets are μ-measurable also. Now by Lemma 3.5

$$\int_K \int_K \int_{a\in B_r} \frac{da\, d\mu(\mathbf{i})\, d\mu(\mathbf{j})}{|S_a(\mathbf{i}) - S_a(\mathbf{j})|^t} \leq c \int_K \int_K \phi^t(T_{\mathbf{i}\wedge\mathbf{j}})^{-1} d\mu(\mathbf{i}) d\mu(\mathbf{j}) \leq c \sum_{\mathbf{p}\in\mathcal{A}^F} \sum_{\mathbf{i}\neq\mathbf{j}} \phi^t(T_{\mathbf{p}})^{-1} \mu(I_{\mathbf{p},\mathbf{i}}) \mu(I_{\mathbf{p},\mathbf{j}})$$

$$\leq c \sum_{\mathbf{p}\in\mathcal{A}^F} \phi^t(T_{\mathbf{p}})^{-1} \mu(I_{\mathbf{p}})^2 \underbrace{\leq}_{\text{By (3.2)}} cc_1 \sum_{r=1}^{\infty} \sum_{\mathbf{p}\in\mathcal{A}^r} \phi^t(T_{\mathbf{p}})^{-1} \phi^s(T_{\mathbf{p}}) \mu(I_{\mathbf{p}}) \underbrace{\leq}_{\text{See † below}} cc_1 \sum_{r=1}^{\infty} \sum_{\mathbf{p}\in\mathcal{A}^r} b^{r(s-t)} \mu(I_{\mathbf{p}})$$

$$\leq cc_1 \mu(K) \sum_{r=1}^{\infty} b^{r(s-t)} < \infty$$

For † notice that for any $u, v > 0$, $\phi^{u+v}(T) \leq \phi^u(T) b^v$ so $\phi^u(T) \geq \phi^{u+v}(T)/b^v$, where b is the largest singular value of T. Hence $\phi^t(T) \geq \phi^{t+(s-t)}(T) b^{t-s} = \phi^s(T) b^{t-s}$ and $\phi^s(T) \phi^t(T)^{-1} \leq b^{s-t}$.
So by the previous lemma, $\dim \Lambda(A) \geq t$ for almost all $a \in B_r$, and as r is arbitrary, $\dim \Lambda(a) \geq t$ for almost all $a \in \mathbb{R}^{nk}$. This is true for all $t \notin \mathbb{N}$ with $t < \min\{n, d(T_1, \ldots, T_k)\}$ so the theorem follows from the lower bound in Lemma 3.7. □

Example 3.10. *Consider the self-affine set given by the transformations*

$$T_1 = T_2 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \ T_3 = \frac{1}{2}\begin{pmatrix} \cos(\frac{\pi}{4}) & \sin(\frac{\pi}{4}) \\ -\sin(\frac{\pi}{4}) & \cos(\frac{\pi}{4}) \end{pmatrix}$$

$$a_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ a_2 = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}, \ a_3 = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \end{pmatrix}$$

The singular values of the T_i are $1/2$ for all of them, which is too large for a calculation by Theorem 3.9, but

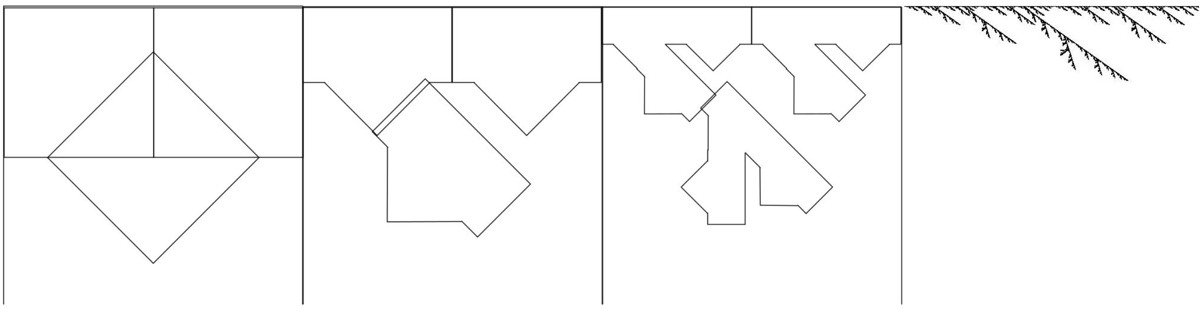


Figure 10: The first three stages in the construction of the invariant set Λ of Example 3.10 starting from the compact set $[0,1] \times [0,1]$.

we can get around this by considering the iterated function system $\{S_i \circ S_j : 1 \leq i, j \leq 3\}$ which determines the same invariant set. The singular values of these functions are then all $1/4$. Hence

$$\tau(s) = \lim_{k\to\infty} \left(9^k \frac{1}{4^{ks}}\right)^{1/k} = 9 \cdot 4^{-s}$$

and by Theorem 3.9, almost surely $\dim_{\mathcal{H}}(\Lambda) = \log 9/\log 4$.

Example 3.11. *The following pictures are all given by the following linear maps with different choices of translations.*

$$T_1 = T_2 = \frac{3}{10}\begin{pmatrix} 0.866 & -0.5 \\ 0.5 & 0.866 \end{pmatrix}, \ T_3 = T_4 = \frac{3}{10}\begin{pmatrix} 0.866 & 0.5 \\ 0.5 & -0.866 \end{pmatrix}$$

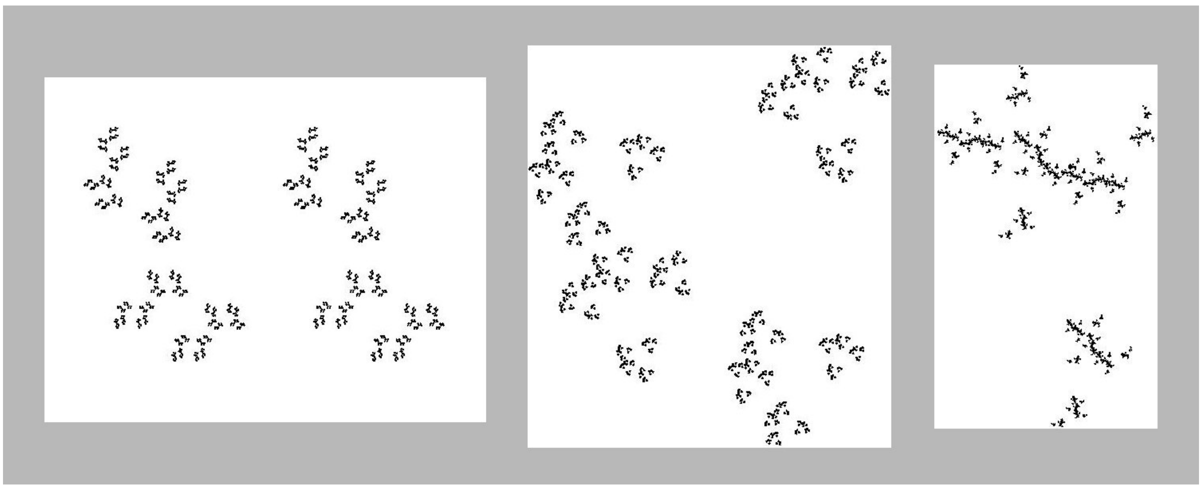


Figure 11: Three invariant sets generated by the same linear maps with different choices of translation. The Hausdorff dimension of these invariant sets is almost always ≈ 1.1514 by Theorem 3.9.

3.3 Remarks on Falconer's Theorem

3.3.1 A counter-example for $\|T_i\| < 1/2 + \varepsilon$

Solomyak [12] provides an example to show that Theorem 3.9 fails for $\|T_i\| < 1/2 + \varepsilon$. If $T_1 = T_2 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \lambda \end{pmatrix}$ then $d(T_1, T_2) = 2 - \log(1/\lambda)/\log 2$. Now if a_1, a_2 don't lie on the same horizontal or vertical line, then it has been shown by Przytycki and Urbański[19], that $\dim_{\mathcal{H}} \Lambda(a) < 2 - \log(1/\lambda)/\log 2$ if λ is a reciprocal of a PV number. Recalling that a PV number is an algebraic integer (a root of a monic polynomial in $\mathbb{Z}$) is an algebraic integer larger than 1 whose algebraic conjugates all have modulus less than 1. We can find a sequence of PV numbers approaching 2 from below, giving a sequence of $\lambda > 2$ but arbitrarily close, this completes the example.

3.3.2 An alternative condition to $\|T_i\| < 1/2$

Solomyak [12] gives an alternative condition to $\|T_i\| < 1/2$ in the self-affine case, for which Theorem 3.9 still holds. Let

$$\mathcal{B} = \left\{ \sum_{j=0}^{\infty} f_j x^j \ : \ f_j \in \{-1, 0, 1\} \right\} \text{ and } \mathcal{D} = \{\lambda \in \mathbb{D} : \forall f \in B_r \, , \, f \neq 0 \, , \, f(\lambda) \neq 0\}$$

where $\mathbb{D} \subset \mathbb{C}$ is the open unit disc. Solomyak then proves that the result holds if $T_i = T$ for all i, and all eigenvalues of T lie in $\mathcal{D}$. The proof uses techniques from complex analysis to give an alternative proof of Lemma 3.5 with the above hypothesis.

3.3.3 Self-Affine Sets in $\mathbb{R}^2$

Heuter and Lalley[14] give the follows list of conditions for iterated function systems in $\mathbb{R}^2$ which guarantee that the point a is not in the set of measure zero for which Falconer's theorem fails.

1. (Contractivity) $\|T_i\| < 1$ for all i
2. (Distorsion) $\alpha_1^2(T_i) < \alpha_2(T_i)$ for $\alpha_1(T_i) \geq \alpha_2(T_i)$ the lengths of the semi-axis of T_i
3. (Separation) Let Q_2 be the closed second quadrant $\mathbb{R}$ 2 (0,0); then the sets $T(Q_i)$ are pairwise disjoint subsets of the interior of Q_i.
4. (Closed set condition) There exists a bounded open set O such that $\overline{A}_1(O), \ldots, \overline{A}_k(O)$ are pairwise disjoint closed subsets of O.

Their proof uses techniques from ergodic theory to construct a suitable measure which can be used alongside thermodynamic formalism.

3.3.4 The Hausdorff Dimension of Exceptional Sets

Falconer and Miao[15] estimate the Hausdorff dimension of the so-called exceptional sets

$$E(s) = \left\{ a \in \mathbb{R}^{nk} \ : \ \dim_{\mathcal{H}} \Lambda(a) < s \right\}$$

to give the following

Theorem 3.12. *If $\|T_i\| < \frac{1}{2}$ for all i, $0 < s \leq \min\{n, d(T_1, \ldots, T_k)\}$ and for any $s > 0$*

$$q_s = nk - \frac{\log \left(\lim_{k \to \infty} \left(\sum_{\mathbf{i} \in K} \phi^s(T_{\mathbf{i}}) \right)^{\frac{1}{k}} \right)}{\log \lambda}$$

Then

$$\dim_{\mathcal{H}} E(s) \leq \max\{nk - (n - s), q_s\}$$

Their proof revolves around a clever alteration to the definition of $\mathcal{M}^s$ and uses many of the same techniques as found in Section 3.

3.3.5 Upper Triangular matrices

In general, the formula of theorem 3.9 can be very hard to calculate. There is a simplification however in the case where all the linear maps are given by upper triangular matrices. If we write

$$T_i = \begin{pmatrix} t_1^i & t_{12}^i & \cdots & t_{1n} \\ 0 & t_2 & \cdots & t_{2n}^i \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & t_n^i \end{pmatrix}$$

With the notation that $\}j_1, \ldots, j_k\{$ represents a set of m distinct integers in $\{1, \ldots, m\}$, then Falconer and Miao have shown in [18] that the following holds. Falconer and Miao use the exterior algebra to give results about the connection between the singular value function and minors of upper triangular matrices.

Theorem 3.13.

$$d(T_1, \ldots, T_m) = \left\{ s \; : \; \max_{\substack{\}j_1,\ldots,j_{m-1}\{ \\ \}j'_1,\ldots,j'_m\{ \\ m-1<s\leq m}} \left\{ |t^1_{j_1} \cdots t^1_{j_{m-1}}|^{m-s} |t^1_{j'_1} \cdots t^1_{j'_m}|^{s-m+1} + |t^N_{j_1} \cdots t^N_{j_{m-1}}|^{m-s} |t^N_{j'_1} \cdots t^N_{j_m}|^{s-m+1} \right\} \right\}$$

4 Quasi-Self-Similar Sets

4.1 Implicitly Calculating the Hausdorff Dimension

Previously we concentrated on methods of estimating the Hausdorff dimension directly, the following results follow a different tack, giving implicit conditions for the Hausdorff and Box counting dimensions to coincide. This can be useful very given that the Hausdorff dimension may be very tricky to calculate where as the box dimension is often significantly easier. The idea is based on what Falconer calls *quasi-self-similarity* or *re-normalization*. Roughly speaking we ask for a condition on a set F that says "large" neighbourhoods look very similar to "small" neighbourhoods of F. Both these theorems were first proved by Falconer in [21], improving apon the results of Laughlin in [25], and will turn out to have many useful applications.

Theorem 4.1. *If F is a set with $\dim_{\mathcal{H}}(F) = s$ and constants $a, r_0 > 0$ such that for any $N \subseteq F$ with $|N| < r_0$ there exists a mapping $\phi : N \to F$ satisfying*

$$ad(x, y) \leq |N| d(\phi(x), \phi(y)) \text{ for all } x, y \in N \tag{4.1}$$

then $\dim_{\mathcal{H}}(F) = \overline{\dim}_B(F) = \underline{\dim}_B(F)$.

Proof. Suppose $s > \dim_{\mathcal{H}}(F)$ (so $\mathcal{H}^s(F) = 0$). Now choose an open cover $\{U_i\}_{i=1}^n$ of F by sets of diameter at most $\min\{a/2, r_0\}$ and such that $\sum_{i=1}^n |U_i|^s < a^s$. By assumption there exist maps $\phi_i : U_i \to F$ such that $ad(x, y) \leq |U_i| d(\phi_i(x), \phi_i(y))$ for all $x, y \in U_i$. Consider the inverse maps ϕ_i^{-1} and model their action using the shift space in the following way: Let $\mathcal{A} = \{1, \ldots, n\}$ and define $U : \mathcal{A}^F \to F$ by $(i_0, \ldots, i_q) \mapsto \phi_{i_0}^{-1} \circ \cdots \circ \phi_{i_q}^{-1}(F)$. Note that $U(\mathbf{i})$ may be empty if the domain and codomain of $\phi_{i_j}^{-1}$ and $\phi_{i_{j+1}}^{-1}$ and do not overlap for some $0 \leq j \leq q - 1$. Now if $x, y \in U((i_0, \ldots, i_q))$ then by (4.1)

$$d(x, y) \leq a^{-1} |U_{i_q}| d(\phi_{i_q}(x), \phi_{i_q}(y)) \leq \cdots \leq a^{-q} |U_{i_q}| \cdots |U_{i_0}| d(\phi_{i_0} \circ \cdots \circ \phi_{i_q}(x), \phi_{i_0} \circ \cdots \circ \phi_{i_q}(y))$$

Thus $|U((i_0, \ldots, i_q))| \leq a^{-q} |U_{i_0}| \cdots |U_{i_q}| F|$
Next let $0 < b < \min a^{-1}|U_i|$ and notice $\max a^{-1}|U_i| < 1/2$ (we chose the U_i initially so this would be true), fix $0 < \varepsilon < |F|$ and for each $(i_0, \ldots, i_q, \ldots)$, curtail the sequence at the least value q such that

$$b\varepsilon < \left(a^{-1}|U_{i_0}|\right) \left(a^{-1}|U_{i_1}|\right) \cdots \left(a^{-1}|U_{i_1}|\right) |F| \leq \varepsilon$$

and hence

$$b^s \varepsilon^s < \left(a^{-s}|U_{i_0}|^s\right) \left(a^{-s}|U_{i_1}|^s\right) \cdots \left(a^{-s}|U_{i_q}|^s\right) |F|^s \leq \varepsilon^s \tag{4.2}$$

Let S be the set of such curtailed sequences. Recalling that at the beginning we asked that $\sum_{i=1}^n |U_i|^s < a^s$, ie. that $\sum_{i=1}^k \left(a^{-1}|U_i|\right)^s < 1$, combining this with (4.2) gives

$$b^s \varepsilon^s |S| < \sum_{i_1, \ldots, i_q \in S} \left(a^{-1}|U_{i_0}|\right)^s \left(a^{-1}|U_{i_1}|\right)^s \cdots \left(a^{-1}|U_{i_1}|\right)^s |F|^s < |F|^s$$

Hence $|S| < |F|^s (b\varepsilon)^{-s}$ and F is covered by at most $|F|^s (b\varepsilon)^{-s}$ sets of diameter at most ε. Using the notation of Section 1.3 where $N_\varepsilon(F)$ denotes the smallest number of sets of diameter at most δ needed to cover F, this gives $N_\varepsilon(F) < |F|^s (b\varepsilon)^s$ so

$$\overline{\dim}_B(F) = \limsup_{\varepsilon \to 0} \frac{\log N_\varepsilon(F)}{-\log \varepsilon} \leq s$$

Hence $\overline{\dim}_B(F) \leq \dim_{\mathcal{H}}(F)$ and since it is always the case that $\dim_{\mathcal{H}}(F) \leq \underline{\dim}_B(F) \leq \overline{\dim}_B(F)$ this gives the result. □

We now prove an analagous result, whereas the previous Theorem required a mapping from some small portion of F onto F, this requires a mapping going in the other direction but satisfying similar properties. The following Theorem also allows us to deduce the useful fact that $\mathcal{H}^s(F) < \infty$. Interestingly the two are proved in very different ways, the previous proof using the condition (4.1) to obtain a bound on $\dim_B(F)$ using an open cover that was 'small' in the sense of Hausdorff dimension. This next theorem assumes that the Hausdorff and Box counting dimensions differ and uses (4.3) this to construct a measure on F. Notice also that the argument at the end is very similar to that used in the Mass Distribution Principle (Lemma 1.16).

Theorem 4.2. *If F is a set with $\dim_{\mathcal{H}}(F) = s$ and constants c, r_0 such that for any ball B in F of radius $r < r_0$ there exists a mapping $\psi : F \to B$ satisfying*

$$crd(x,y) \leq d(\psi(x), \psi(y)) \text{ for all } x, y \in F \tag{4.3}$$

then $\dim_{\mathcal{H}}(F) = \overline{\dim}_B(F) = \underline{\dim}_B(F)$ and $\mathcal{H}^s(F) < \infty$.

Proof. Recall the notation from section 1.3 that $\tilde{N}_\varepsilon(F)$ is the largest possible number of disjoint balls of radius ε and centers in F. We assume that there exists some $\varepsilon < \min\{c^{-1}, r_0\}$ such that $n = \tilde{N}_\varepsilon > c^{-s}\varepsilon^{-s}$. Then choose a $t > s$ such that

$$\tilde{N}_\varepsilon(F) > c^{-t}\varepsilon^{-t} \tag{4.4}$$

Let $B_1, \ldots, B_n$ be closed disjoint balls of radius ε with centers in F and let $\delta = \min_{i \neq j} d(B_i, B_j) > 0$. By assumption there exist mappings $\psi_i : F \to B_i$ for each $1 \leq i \leq n$ which satisfy $c\varepsilon d(x,y) \leq d(\psi_i(x), \psi_i(y))$ for all $i, j \in F$. We regard these ψ_i like an iterated function system (although they may not be contractions). Let $\mathcal{A} = \{1, \ldots, n\}$ and define a map η on $\mathcal{A}^F$ by

$$\eta((i_0, \ldots, i_q)) = \phi_{i_0} \circ \phi_{i_1} \circ \cdots \circ \phi_{i_q}(F)$$

Notice that $d(\eta((i_0, \ldots, i_q)), \eta((j_0, \ldots, j_q))) \geq (c\varepsilon)^{m-1} d(B_{i_m}, B_{j_m}) \geq (c\varepsilon)^q \delta$ where m is the position in which $(i_0, \ldots, i_q)$ and $(j_0, \ldots, j_q)$ first differ.
Define a measure μ on F by first defining it as a premeasure on the subsets $\eta((i_0, \ldots, i_q))$ via $\mu\left(\eta((i_0, \ldots, i_q))\right) = n^{-q}$ and extending this via Method I (Lemma 1.2) to a measure on F. Now for any subset $U \subseteq F$ with $|U| < c \leq \delta$, let q be the least integer such that

$$(c\varepsilon)^q \delta > |U| \geq (c\varepsilon)^{q+1}\delta \tag{4.5}$$

Now U intersects at most one B_i, so

$$\mu(U) \leq n^{-q} \underbrace{\leq}_{\text{by (4.4)}} (c\varepsilon)^{qt} \underbrace{\leq}_{\text{by (4.5)}} (c\varepsilon\delta)^{-t}|U|^t$$

Now suppose F is covered by an open cover $\{U_i\}_{i=1}^\infty$ with $|U_i| < c\varepsilon\delta$ so

$$1 = \mu(F) \leq \sum_{i=1}^\infty \mu(U_i) \leq (c\varepsilon\delta)^{-t} \sum_{i=1}^\infty |U_i|^t$$

This implies $\mathcal{H}^t(F) > 0$, which is a contradiction since $t > s$ and $\dim_{\mathcal{H}}(F) = s$. Hence our original assumption was incorrect, and for small ε, $\tilde{N}_\varepsilon(F)\varepsilon^s \leq c^{-s}$ so

$$\overline{\dim}_B = \limsup_{\varepsilon \to 0} \frac{\log \tilde{N}_\varepsilon(F)}{-\log \varepsilon} \leq s$$

To see that $\mathcal{H}^s(F) < \infty$, notice that given a maximal (finite) set of disjoint balls with centers in F if we take the same centers but with twice the radius they form a cover of F. □

It is interesting to compare this theorem to the results about iterated function systems we proved earlier in section 2, without the open set condition we can still say that the Hausdorff and box-counting dimensions coincide and that the s-dimensional Hausdorff measure is positive and finite, without being able to actually calculate the Hausdorff dimension.

Theorem 4.3. *Given an IFS of similarities $\{S_i\}_{i=0}^m$ with ratios $0 < c_i < 1$ and invariant set Λ then $\dim_{\mathcal{H}} \Lambda = \overline{\dim}_B \Lambda = \underline{\dim}_B \Lambda$ and $\mathcal{H}^s(\Lambda) < \infty$ where $s = \dim_{\mathcal{H}}(\Lambda)$*

Proof. Let $z \in \Lambda$ and $\mathbf{i} = (i_0, \ldots, i_n, \ldots)$ be the sequence such that $S(\mathbf{i}) = z$. Now Given any $r < |F|$ there exists a least integer q such that

$$c_- r \leq c_{i_0} \cdots c_{i_q}|F| \leq r$$

where $c_- = \min_i c_i$. Now, writing $\psi(x) = S_{i_0} \circ \cdots \circ S_{i_q}(x)$ we may combine the above with the fact that ψ is a similarity to get

$$ard(x,y) \leq d(\psi(x), \psi(y))|F| \leq rd(x,y)$$

Now let $x = \psi^{-1}(z)$ and it is clear that ψ maps F into $B(z,r)$. We may now apply Theorem 4.2 to obtain $\dim_{\mathcal{H}}(\Lambda) = \underline{\dim}_B(\Lambda) = \overline{\dim}_B(\Lambda)$ and $\mathcal{H}^s(\Lambda) < \infty$. □

4.2 Dynamical Repellers

An interesting example of the use of the previous theorems is the following. In this section we will be working with Riemannian manifolds, although the reader unfamiliar with such things can simply think of this as manifolds M and N as $\mathbb{R}^n$ for some n and the pull-back of a map $f : M \to N$ as the usual derivative.

Definition 4.4. *A map $T : M \to M$ is $C^{1+\alpha}$ if its pull-back f_* satisfies the Hölder condition of exponent α (see Lemma 1.9).*

Definition 4.5. *Given $f : M \to M$ a $C^{1+\alpha}$ map, we say a compact set $J \subseteq M$ is a (mixing) repeller for f if it satisfies*

1. *f is expanding on J ie. there exist constants $c > 0$ and $\alpha > 0$ such that $\|(D_x f^n)u\| \geq c\alpha^n \|u\|$ for all $x \in J$, $n \geq 1$ and $u \in T_x M$.*
2. *J is completely invariant under f, ie $f(J) = J$ and $f^{-1}(J) = J$.*
3. *f is mixing on J (for any open set U such that $U \cap J \neq \varnothing$ there exists an n such that $J \subseteq f^n(U)$).*

Recall that a map $f : M \to N$ is conformal if $g_M = \lambda f^* g_N$ where f^* is the pull-back of f, λ is a positive function on M and g_M and f_N are the riemannian metrics on M and N respectively. For the reader unfamiliar with Riemannian manifolds, when working with $\mathbb{R}^n$ this is equivalent to the regular notion of conformality. We are now in a position to apply the Theorems of the previous section to obtain the following result.

Theorem 4.6. *If f is a $C^{1+\alpha}$ conformal mapping with mixing repeller J and $s = \dim_{\mathcal{H}}(J)$, then $\overline{\dim}_B(J) = \underline{\dim}_B(J) = s$ and $0 < \mathcal{H}^s(J) < \infty$. In fact the conditions of Theorem 4.1 and 4.2 hold.*

A proof of the above theorem can be found in [21]

4.3 Locally Expanding Maps

In this section we will be working with expanding maps $T : X \to X$, we will see that for a specific class of these maps it is possible to define local inverses such that the collection of these local inverses forms an iterated function scheme. The next two definitions are crucial for this idea.

Definition 4.7. *Let X be a compact space.*

1. *A map $T : X \to X$ is $C^{k+\alpha}$ if its k^{th}partial derivatives exist and satisfy the Hölder condition of exponent α (see Lemma 1.9). If $k = 0$ we will abbreviate this to C^α which is the space of α-Hölder continuous functions.*
2. *A map is locally expanding there exist constants $c > 0$ and $\beta > 0$ such that $\|(D_x f^n)u\| \geq \beta^n \|u\|$ for all $x \in J$, $n \geq 1$ and $u \in T_x M$.*

Definition 4.8. *A* Markov Partition *of $T : X \to X$ is a finite collection $\mathcal{P} = \{P_i\}_{i=0}^n$ of closed sets which satisfy*

1. *$\cup_{i=1}^n P_i = X$*
2. *Each P_i is proper (the closure of the interior of P_i is P_i itself).*
3. *TP_i is the union of elements of $\mathcal{P}$.*
4. *$T|_{P_i} : P_i \to T(P_i)$ is a local homeomorphism.*

Markov Partitions are the key to the local inverses mentioned above, the fourth part of the definition implies that we can define local inverses T_i from $T(P_i) \to P_i$, and the fact $T(P_i)$ is a union of sets in $\mathcal{P}$ means that when we try and form an iterated function scheme by considering sets of the form $T_{i_0} \circ \cdots \circ T_{i_n}(X)$, it is easy to check if the maps are well defined.

Example 4.9. *Consider the map $f : S^1 \to S^1$ taking $z \mapsto z^2$ (where S^1 is viewed as a subset of $\mathbb{C}$). f is a $C^{1+\alpha}$ locally expanding map:*

- *S^1 is compact*
- *C^1 is obvious and $f'(z) = 2z$ which is Lipschitz (Hölder continuous with $\alpha = 1$), so f is C^{1+1}*
- *$f^n(z) = z^{2n}$, so $D_z f^n = 2nz^{2n-1}$ and locally expanding is clear.*

We may define a Markov Partition by taking $P_1 = \{e^{i\theta} : \theta \in [0, \pi]\}$ and $P_2 = \{e^{i\theta} : \theta \in [\pi, 2\pi]\}$, with local inverses given by

$$T_1 : S^1 \to P^1 \ , \ T_1 : e^{i\theta} \mapsto e^{i\theta/2} \text{ and } T_2 : S^1 \to P_2 \ , \ T_1 : e^{i\theta} \mapsto e^{i\theta/2+\pi/2}$$

These are both similarities with contraction factor $1/2$. Moreover it is easy to see they form an iterated function system with invariant set $\Lambda = S^1$ and satisfying the open set condition with $O = S^1 \setminus \{1\}$, hence we can use Theorem 2.4 to calculate $\dim_{\mathcal{H}}(S^1) = s$ where s is the unique number such that $2(1/2)^s = 1$, ie. $\dim_{\mathcal{H}}(S^1) = s = 1$.

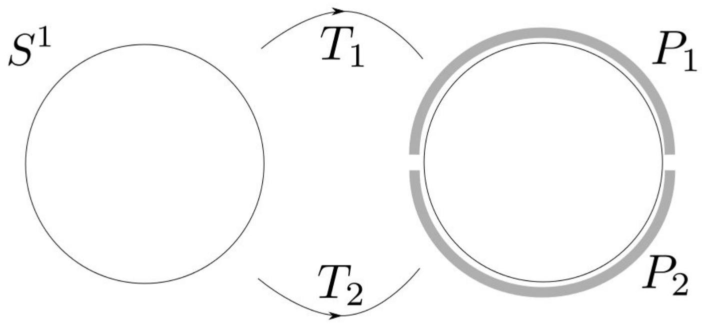


Figure 12: Example 4.9

The main reason we use $C^{1+\alpha}$ locally expanding maps is because of the following Lemma, a proof of which can be found in [26].

Lemma 4.10. *If $T : X \to X$ is a $C^{1+\alpha}$ locally expanding map then there exists a Markov Partition*

We may use these Markov Partitions to define the set X in terms of Iterated Function Schemes. Consider the local inverses of the maps $T|_{P_i} : P_i \to T(P_i)$ which we denote by T_i. These maps are contractions since $T_i'(z) \leq 1/\lambda$ implies $d(T_i(x), T_i(y)) \leq (1/\lambda)d(x, y)$ hence we can view $\{T_1, \dots, T_n\}$ as an iterated function scheme. Ideally we want to recover X as the invariant set of the iterated function scheme. In example 4.9 the iterated function scheme was easy to see since $T|_{P_i}$ had codomain $S^1 = P_1 \cup P_2$ for every i, $\cup_i T_i(S^1) = S^1$ and in general

$$\bigcup_{(i_0,\dots,i_q)\in\mathcal{A}^q} T_{i_0} \circ \cdots \circ T_{i_q}(S^1) = S^1$$

Where $\mathcal{A} = \{0, 1\}$. So the invariant set will be exactly Λ. Of course it is not always simple, as the domains and codomains of our local inverses may not line up as in the next example.

Example 4.11. *We can alter example 4.9 by choosing the Markov partition $P_1 = \{e^{i\theta} : \theta \in [0, \pi/2]\}, P_2 = \{e^{i\theta} : \theta \in [\pi/2, \pi]\}, P_3 = \{e^{i\theta} : \theta \in [\pi, 3\pi/2]\}, P_4 = \{e^{i\theta} : \theta \in [3\pi/2, 2\pi]\}$. Now we have that*

$$T(P_1) = P1 \cup P_2\ ,\ T(P_2) = P_3 \cup P_4\ ,\ T(P_3) = P_1 \cup P_2\ ,\ T(P_4) = P_3 \cup P_4$$

giving local inverses

$$T_1 : P_1 \cup P_2 \to P_1\ ,\ T_2 : P_3 \cup P_4 \to P_2\ ,\ T_3 : P_1 \cup P_2 \to P_3\ ,\ T_4 : P_3 \cup P_4 \to P_4$$

each a similarity with contraction factor 1/2. Now it only makes sense to consider $T_i \circ T_j$ when their codomains and domains line up, so we get the iterated function scheme associated to the subshift of finite type given by

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

This matrix has largest real eigenvalue 2 so appealing to Theorem 2.18 we get that $\dim_{\mathcal{H}}(S^1) = 1$ again.

So in general given a $C^{1+\alpha}$ locally expanding map $T : X \to X$ with Markov partition $\mathcal{P} = \{P_i\}_{i=1}^n$ we will get an iterated function scheme of local inverses T_i associated to a subshift of finite type given by a matrix A, where $A_{i,j} = 1$ if and only if the domain of T_i is in the codomain of T_j. This will then give us an invariant set $\Lambda = X$. To visualise this more formally, let $\mathcal{A}_A$ be the subshift of finite type associated the matrix A then $\cup_{i,j\in\mathcal{A}^2} T_i(U_j) = X$ and more generally

$$\bigcup_{(i_0,\dots,i_q)\in\mathcal{A}_A^q} T_{i_0} \circ \cdots \circ T_{i_{q-1}}(U_{i_q}) = X$$

We can then write this in a similar way to that in previous sections by setting

$$T((i_0, \dots, i_q)) = T_{i_0} \circ \cdots \circ T_{i_{q-1}}(U_q) \tag{4.6}$$

4.4 Bowen's Formula

Throughout this section, $T : X \to X$ is a $C^{1+\alpha}$ conformal expanding map, $\mathcal{P} = \{P_i\}_{i=1}^k$ is a Markov Partition as given by Lemma 4.10 and T_i are the local inverses of T on the elements P_i of $\mathcal{P}$. The following result is due to Bowen and Ruelle, the proof of the result is long and complex so I will just give a sketch of the main ideas. A full proof can be found in [27].

Theorem 4.12. Bowen-Ruelle Formula *Let $T : X \to X$ be a $C^{1+\alpha}$ conformal expanding map then there is a unique solution to $P(-s\log|T'|) = 0$ which occurs when $s = \dim_{\mathcal{H}}(X) = \dim_B(X)$*

Here $P(f)$ denotes the topological pressure function with respect to T, defined for any continuous $f : X \to \mathbb{R}$

$$P(f) = \lim_{n\to\infty} 1/n \log \sum_{x\in \mathrm{Fix}(T^n)} \exp\left(\sum_{i=0}^{n-1} f(T^i(x))\right)$$

The idea is to use the sets of the Markov Partition $\mathcal{P} = \{P_i\}_{i=1}^n$ as a cover to estimate the Hausdorff dimension, however since the P_i are closed we first choose open sets $U_i \supset P_i$ such that the difference between the U_i and P_i is very small. Next, using the Hölder continuity and conformality of T we can prove the following bound where $\mathcal{A} = \{1, \ldots, n\}$.

$$c_1 \le \frac{\sum\limits_{(i_0,\ldots,i_q)\in\mathcal{A}^q} |T((i_0,\ldots,i_q))|^t}{\sum\limits_{(i_0,\ldots,i_q)\in\mathcal{A}^q} \left|(T_{i_0}\circ\cdots\circ T_{i_q})'(x)\right|^t} \le c_2$$

for all $t > 0$, all x and some c_1, c_2, positive constants and using the notation for T from (4.6). This allows us to translate statements about $|T(i_0,\ldots,i_1)|$ into statements about $|(T_{i_0}\circ\cdots\circ T_{i_q})'|$.
Next we introduce the *Ruelle Operator* $\mathcal{L}_t$ which acts on $\mathcal{C}^\alpha$, the set of Hölder continuous functions acting on $\mathcal{P}$ as a disjoint union. The Ruelle Operator is defined as

$$\mathcal{L}_t : \omega(x) \mapsto \sum_{i=1}^n |T_i'(x)|^t\, \omega(T_i(x))$$

Using techniques from functional analysis it can be shown that

- $\mathcal{L}_t$ has a maximal positive eigenvalue λ_t isolated away from the rest of the spectrum of $\mathcal{L}_t$ and
- $t \mapsto \lambda_t$ is analytic
- $P(-t\log|T'|) = \log\lambda_t$

Using this final fact we may bound $\inf\{\sum_i |V_i|^t \ : \ V_i \text{ is a } \delta\text{-cover of } X\} \le C\lambda_t^n$ where n is large enough that $|T(\mathbf{i})| \le \delta$ for all $\mathbf{i} \in \mathcal{A}^F$ of length greater than n. From here we can conclude that $\dim_{\mathcal{H}}(X) \le s$ where s is given by Theorem 4.12. To obtain the lower bound, we can use the Ruelle Operator again to give a probability measure on X and then apply the Mass Distribution Principle.

Remark 4.13. *Of great interest is the fact that the map $t \mapsto \lambda_t$ is analytic, as this implies that the function $t \mapsto P(-t\log|T'|)$ is analytic and so if we have analytic family T_r for $r \in (-\varepsilon, \varepsilon)$ satisfying $\partial f/\partial r$ is everywhere non-zero then by the implicit function theorem the function $r \mapsto \dim_{\mathcal{H}}(\Lambda_r)$ is also analytic (where Λ_r is the invariant set of T_r).*

4.5 Iteration of Holomorphic Maps

A fascinating use of Hausdorff dimension comes from looking at sets points which are 'unpredictable' under iteration by holomorphic functions. We will see that for a very small class of these functions, namely the quadratics, this set is uniquely described up to Hausdorff dimension. We will be working with holomorphic functions on the riemann sphere $\hat{\mathbb{C}}$, the spherical metric on $\hat{\mathbb{C}}$ will be denoted by d. Recall that a countable family $\{f_n\}_{n=1}^\infty$ of holomorphic functions is *normal* on a set $U \in \hat{\mathbb{C}}$ if every sequence of functions in that family has a subsequence which converges uniformly (with respect to d) on compact sets to a continuous function on U. We will also need the following

Theorem 4.14. Montel's Theorem *A family of holomorphic maps $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is normal on U if and only if it is locally bounded on U. (Where locally bounded means for given $\varepsilon > 0$ there exists $\delta > 0$ such that $d(z, w) < \delta$ implies $d(f_n(z), f_n(w)) < \varepsilon$ for all $n \in \mathbb{N}$ and all $z, w \in U$).*

A proof of Montel's theorem can be found in Conway[23]. We say that a family is normal at a point z if there exists an open neighbourhood U on which the family is normal.
We will be considering holomorphic functions on $\hat{\mathbb{C}}$, it is easily proved using techniques in complex analysis that the set of holomorphic functions on $\hat{\mathbb{C}}$ is exactly the set of rational functions on $\hat{\mathbb{C}}$ (See Conway[23] for example)

Definition 4.15. *The* Fatou *set of a rational function* $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ *is the set*

$$F(f) = \{z \ : \ \{f^n(z)\}_{n=0}^{\infty} \text{ is a normal family}\}$$

The Julia *set is defined as* $J(f) = \hat{\mathbb{C}} \backslash F(f)$.

Notice that $F(f)$ is open by definition and hence also $J(f)$ is closed. Intuitively we think of the Fatou set as being the set of points which are fairly 'stable' under iteration and the Julia set as the set of points which are 'chaotic' or unpredictable under iteration by f.

Definition 4.16. *A set* $E \subseteq \hat{\mathbb{C}}$ *is* completely invariant *under* f *if* $f(E) = E$ *and* $f^{-1}(E) = E$

Lemma 4.17. $J(f)$ *and* $F(f)$ *are completely invariant under f.*

Proof. For readability, I will write F for $F(f)$ and J for $J(f)$ in this proof. Firstly we show that $f^{-1}(F) \subseteq F$. Let $z_0 \in f^{-1}(F)$ and write $w_0 = f(z_0) \in F$. By the definition of F given by Montel's Theorem we have that given some $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$d(w, w_0) < \delta \Rightarrow \forall n \ d(f^n(w), f^n(w_0)) < \varepsilon$$

Now by continuity of f at z_0 there exists a δ' such that

$$d(z, z_0) < \delta' \Rightarrow d(f(z), f(z_0)) = d(f(z), w_0) < \delta$$

Combining the above we get that $d(f^{n+1}(z), f^{n+1}(z_0)) = (f^n(f(z)), f^n(w_0)) < \varepsilon$, so $\{f^{n+1}\}_{n=1}^{\infty}$ and hence $\{f^n\}_{n=1}^{\infty}$ is normal at z_0, thus giving $f^{-1}(F) \subseteq F$.
For the opposite inclusion let $z_0 \in F(f)$ and $w_0 = f(z_0)$. By definition of F, given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d(z, z_0) < \delta \Rightarrow \forall n \ d(f^n(z), f^n(z_0)) < \varepsilon$$

Now, let $U = \{z : d(z, z_0) < \delta\}$ and consider $f(U)$ which is an open neighbourhood of w_0 (by the Open Mapping Theorem). Next if $w \in f(U)$ then $w = f(z)$ for some $z \in U$ so $d(f^n(w), f^n(w_0)) = d(f^{n+1}(z), f^{n+1}(z_0)) < \varepsilon$ giving $w_0 \in F$ and $F \subseteq f^{-1}(F)$.
Finally since all rational maps on $\hat{\mathbb{C}}$ are surjective (an easy consequence of the fundamental theorem of algebra), $f(f^{-1}(F)) = f(F) = F$. Since we have shown F is completely invariant, J must be also since $F = \hat{\mathbb{C}} \backslash J$. □

The next Theorem provides an insight into the behaviour of f on the Julia set, recall that a function f is said to be *mixing* if for any two open sets U and V there exists some N such that for all $n > N$, $f^n(U) \cap V \neq \varnothing$.

Lemma 4.18. *If* $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ *is a rational map with* $\deg f \geq 2$ *then f is mixing on J*

While this result is not too hard to prove, it would take us a little bit out of our way. The proof can however be found in Beardon[24] Theorem 4.2.5.
We are now going to restrict our attention to polynomials of the form $f_c(z) = z^2 + c$. This may seem overly restrictive at first sight but in fact it allows us to consider all quadratics. This is because, in the language of dynamical systems, every quadratic $q(z)$ is conjugate to $f_c(z)$ for some c. What this means is that given any quadratic q we may choose appropriate values of α, β and c to give a (biholomorphic) map $\pi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ where $\pi(z) = \alpha z + \beta$ and $\alpha \neq 0$ with the property that

$$\pi^{-1} \circ f_c \circ \pi(z) = \alpha z^2 + 2\beta z + \frac{\beta^2 + c - \beta}{\alpha} \ \big(= q(z)\big) \text{ ie. that } \pi \circ f_c = q \circ \pi$$

$$\begin{tikzcd}
\hat{\mathbb{C}} \arrow[r, "f_c"] \arrow[d, "\pi"'] & \hat{\mathbb{C}} \arrow[d, "\pi"] \\
\hat{\mathbb{C}} \arrow[r, "q"'] & \hat{\mathbb{C}}
\end{tikzcd}$$

This conjugacy means that $F(f_c) = \pi(F(q))$, to see this first notice that $\hat{\mathbb{C}}$ is compact, π and π^{-1} are continuous and so π and π^{-1} are uniformly continuous on $\hat{\mathbb{C}}$. Let $z_0 \in F(q)$, and fix $\eta > 0$.

- By uniform continuity of π^{-1}, there exists ε such that $d(y_1, y_2) < \varepsilon$ implies $d(\pi^{-1}(y_1), \pi^{-1}(y_2)) < \eta$
- Since $z_0 \in F(q)$ there is a neighbourhood V on which there exists a $\delta > 0$ such that $d(x_1, x_2) < \delta$ implies $d(q^n(x_1), q^n(x_2)) < \varepsilon$ for all n.
- By uniform continuity of π, there exists a δ' such that $d(y_1, y_2) < \delta'$ implies $d(\pi(y_1), \pi(y_2) < \delta$.

Combining the above three, for any $y_1, y_2 \in \pi^{-1}(V)$ such that $d(y_1, y_2) < \delta'$ we have that

$$d(f_c^n(y_1), f_c^n(y_2)) = d(\pi^{-1} \circ q^n \circ \pi(y_1), \pi^{-1} \circ q_n \circ \pi(y_2)) < \eta \text{ for all n}$$

Hence $\pi^{-1}(z_0) \in F(f_c)$ and $\pi^{-1}(F(q)) \subseteq F(f_c)$. By reversing the above argument we easily obtain $F(q) = \pi^{-1}(F(f_c))$.
Since we are now working with polynomials, it is possible to describe the Julia Set in simpler terms. The set $\{z : f^k(z) \nrightarrow \infty\}$ mentioned below is often called the *filled-in Julia set.*

Lemma 4.19. *If f is a polynomial then $J(f) = \partial K$ where $K = \{z : f^k(z) \nrightarrow \infty\}$*

Proof. If $z \in \delta K$ then in any open neighbourhood about z there are points $w_1 \in K$ and $w_2 \notin K$ such that $f^n(w_1) \to \infty$ as $n \to \infty$ and $f^n(w_2) \nrightarrow \infty$ as $n \to \infty$. Hence there is no open neighbourhood about z on which any subsequence of $\{f^n\}_{n=1}^{\infty}$ is uniformly convergent and $z \in J(f)$. Conversely if $z \notin \delta K$ then either $z \in K^{\circ}$ or $z \in \hat{\mathbb{C}} \setminus \overline{K}$. If $z \in K^{\circ}$ then there is an open neighbourhood U on which $\{f^n\}_{n=1}^{\infty}$ is locally bounded and hence by Montel's Theorem $\{f^n\}_{n=1}^{\infty}$ is normal at z and $z \notin J(f)$. If $z \in \hat{\mathbb{C}} \setminus \overline{K}$, then there is an open set U on which $f^k(x) \to \infty$ for all $x \in U$, note that this is not quite strong enough, we need that f^n converges uniformly on compact sets in U.
Claim: There exists an r such that $|f^k(z)| \geq r$ for some k implies that $|f^{k+l}(z)| \geq 2^l r$. If we assume the claim then since $f^n(z) \to \infty$ there exists a k such that $|f^k(z)| \geq r$ so by continuity of f^k there is an open neighbourhood V of z such that $|f^k(x)| \geq r$ for all $x \in V$ and the claim gives that f^k converges uniformly to ∞ on V which completes the proof.
Proof of Claim: Write $f(z) = \sum_{i=0}^{n} a_i z^i$ with $a_n \neq 0$, then we may choose r sufficiently large that

$$|z| \geq r \Rightarrow \frac{|a_n||z|^n}{2} \geq 2|z| \text{ and } \frac{|a_n||z|^n}{2} \geq \sum_{i=0}^{n-1} |a_i||z|^i$$

So $|z| \geq r$ then implies that

$$|f(z)| \geq |a_n||z|^n - \sum_{i=0}^{n-1} |a_i||z|^i \geq \frac{|a_n||z|^n}{2} \geq 2|z|$$

Finally, applying this inductively with the initial condition that $|f^k(r)| \geq z$ we get that $|f^{k+l}(z)| \geq 2^k|z| \geq 2^k r$. □

This next Lemma gives us an intriguing third alternative definition of the Julia set and is essential for the proof of the main Theorem of this section.

Lemma 4.20. *If ω is an attractive fixed point of a polynomial f, then $\partial A(\omega) = J(f)$, where $A(\omega)$ is the* basin of attraction *of ω defined as $A(\omega) = \{z : f^k(z) \to \omega$ as $k \to \infty\}$.*

Proof. If $z \in \partial A(\omega)$ but $z \notin J(f)$ then there is an open neighbourhood U about z on which $\{f^n\}_{n=1}^{\infty}$ is normal. $U \cap A \neq \varnothing$ so $\{f^n\}$ converges to the constant function ω on $U \cap A$ and hence converges to the constant function ω on U by the Identity Theorem (see Conway[23]). Hence $U \subseteq A(\omega)$ which is a contradiction.
For the converse, let $z \in J(f)$, so $f^k(z) \in J(f)$ (follows immediately from Lemma 4.19). Now, if $f^k(z) \to \omega$ then $\omega \in J(f)$ since $J(f)$ is closed, but $A(\omega)$ is an open set about ω so by Lemma 4.19 intersects $\hat{\mathbb{C}} \setminus K = \{z : f^k(z) \to \infty\}$ which is a contradiction. Hence $z \notin A(\omega)$. However, if U is a neighbourhood of z then since f is mixing (Lemma 4.18) there exists a k such that $f^k(U) \cap A(\omega) \neq \varnothing$. So points arbitrarily close to z iterate to ω and so $z \in \overline{A(\omega)}$ and hence $z \in \partial\omega$. □

Notice that when c is zero, 0 is an attractive fixed point and $J(f_c)$ is the unit circle, since $f_0(z) \to 0$ if $|z| < 1$ and $f_0(z) \to \infty$ if $|z| > 1$. When c moves away from zero but is still small, then $\omega = 1/2(1 - \sqrt{1-4c})$ is a fixed point and we would intuitively expect $J(f_c)$ to distort away from a circle, as the next theorem shows, this is exactly what happens. See figure 13 for a graphical demonstration of this.

Theorem 4.21. *If $|c| < 1/4$ then $J(f_c)$ is a simple closed curve.*

Proof. Let C_0 be the curve $|z| = 1/2$, notice that this surrounds the attractive fixed point ω of f_c. Next, consider the loop $C_1 = f_c^{-1}(C_0)$ and notice it surrounds C_0 since the branches of the inverse of f_c are $\pm\sqrt{z-c}$ and when $z = 1/2$, $\sqrt{|z-c|} > 1/2$. Now denote the annulus between C_0 and C_1 by A_1 and for each $\theta \in [0, 2\pi)$ associate a curve γ_θ which leaves C_1 perpendicularly at $(1/2)e^{i\theta}$ and hits C_1 perpendicularly at some point which we denote by $\psi_1(\theta)$. Furthermore we choose these curves so that $\psi(\theta)$ is surjective onto C_1. Now define C_2 as $C_2 = f_c^{-1}(C_1)$, this is again a curve containing C_1 and we denote the annulus between C_2 and C_1 by A_2. We can also extend γ_θ by adjoining to it the curve $f_c^{-1}(\gamma_\theta)$ and denoting the point where it touches C_2

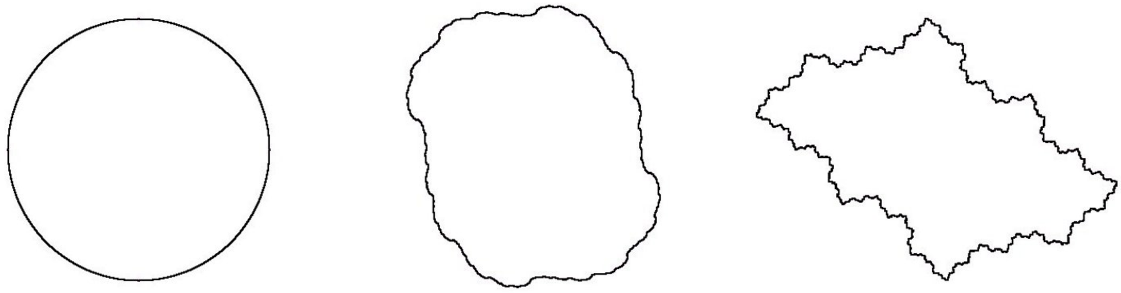


Figure 13: Julia sets $J(f_c)$ where $f_c(z) = z^2 + c$ for the values $c = 0$, $c = 0.15 + 0.15i$, and $c = -0.4 + 0.4i$. Notice the larger the value of $|c|$ the rougher the boundary of $J(f_c)$ becomes. These figures are easily plotted by either computing $\{z : f^n(z) \to \infty\}$ and taking the boundary or finding one value $z \in J(f_c)$ and using the fact that $J(f_c) = \overline{\cup_{k=1}^{\infty} f^{-k}(z)}$ for any $z \in J(f_c)$ (this is easily provable see for example Falconer [1] Corollary 14.8

by $\psi_2(\theta)$. This process can now be continued inductively to get a sequence A_n of annuli between loops C_n and curves γ_θ which touch C_n at $\psi_n(\theta)$.
Now, as $n \to \infty$, the curves C_n will approach $\partial A(\omega)$, and by Lemma 4.20 this is exactly $J(f_c)$. Finally, when z is outside of C_1, $|f_c'(z)| = 2|z| > \alpha > 1$ for some α. It follows that $|(f_c^{-1})'(z)| < 1/\alpha < 1$ for z outside of C_1 and so the length of γ_θ between C_n and C_{n+1} is strictly less that $1/\alpha$ times the length of γ_θ between C_{n-1} and C_n. So the length of γ_θ between C_n and C_{n+1} is decreasing (more than) geometrically to 0 as $n \to \infty$. Since this α was not dependant on θ we can conclude that the $\psi_n(\theta)$ converge uniformly to a continuous function $\psi(\theta) : S^1 \to J(f_c)$.
It remains to show that $J(f_c)$ is a simple curve. Suppose $\psi(\theta_1) = \psi(\theta_2)$ for some $\theta_1 < \theta_2$ and let D be the region bounded by C_0 between θ_1 and θ_2 and the two curves γ_{θ_1} and γ_{θ_2}, now since C_0 doesn't 'escape' from the interior of $J(f_c)$ under iteration and neither does γ_θ, ∂D is bounded under iteration by f_c. Hence by the Maximum Modulus Theorem (see for example Conway[23]) D remains bounded under iteration by f_c. Thus $D \subset \{z : f^k(z) \nrightarrow \infty \text{ as } k \to \infty\}$ and by Lemma 4.19 $D^\circ \cap J = \varnothing$ which in turn implies that $\psi(\theta) = \psi(\theta_1) = \psi(\theta_2)$ for all $\theta \in (\theta_1, \theta_2)$. □

In fact, when $|c| < 1/4$ $J(f_c)$ is what we call a *quasi-circle*, which we will discuss in detail in the next section.

4.6 Classification of Quasi-Circles

Quasi-Circles are a particular class of curves which are homeomorphic to S^1 but also satisfy properties very similar to the quasi-self-similarity properties of Theorem 4.1 and 4.2. From these properties we can deduce the very strong result that any two quasi-circles C_1 and C_2 are Lipschitz equivalent (there exists Lipschitz mappings $C_1 \to C_2$ and $C_2 \to C_1$) if and only if the Hausdorff dimensions of C_1 and C_2 coincide, we can also relate this information back to our results about the Julia set of quadratics from the previous section.

Definition 4.22. *C is a* quasi-self-similar circle *or* quasi-circle *if*

1. *C is homeomorphic to* S^1
2. *There exists* $r_0, a, b > 0$ *such that for every* N *with* $|N| = r < r_0$ *there is a mapping* $\varepsilon : N \cap C \to C$ *such that*
$$ad(x, y) \leq rd(\varepsilon(x), \varepsilon(y)) \leq bd(x, y) \text{ for all } x, y \in N$$
3. *There exists* $r_1, c > 0$ *such that for any ball whose center lies in* C *and has radius* $r < r_1$ *there is a mapping* $\psi : C \to B \cap C$ *satisfying*
$$crd(x, y) \leq d(\psi(x), \psi(y))$$

Notice that the second property incorporates condition (4.1) from Theorem 4.1 and the third condition is exactly (4.3) from Theorem 4.2. This immediately gives us that the Hausdorff and Box dimensions of a quasi-circle coincide and that if $s = \dim_{\mathcal{H}}(C)$ then $\mathcal{H}^s(C) < \infty$.

Lemma 4.23. *1.* $\mathcal{H}^s(C) \in (0, \infty)$
2. $\mathcal{H}^s(I(x, y)) > 0$ *for all* $x \neq y$

3. *For all x, $\mathcal{H}^s(I(x,y)) \to 0$ as $y \to x$*

Proof. 1. This is exactly the remark preceeding this Lemma.

2. Notice that the third property of a quasi-circle implies that the map ψ is both continuous and injective. Given a ball B with radius $r < r_0$, choose an open cover $\{V_i\}_{i=1}^n$ of $B \cap C$ such that $\sum_{i=1}^{\infty} |V_i|^s \leq \mathcal{H}^s(B \cap C) + \varepsilon$. Now $\{\psi^{-1}(V_i)\}_{i=1}^n$ forms an open cover of C and

$$\mathcal{H}^s(C) \leq \sum_{i=1}^{n} |\psi^{-1}(V_i)| \leq \frac{1}{c^s r^s} \sum_{i=1}^{n} |V_i| \leq \frac{1}{c^s r^s} \mathcal{H}^s(C \cap B) + \varepsilon$$

As ε was arbitrary, this gives that $\mathcal{H}^s(B \cap C) \geq c^s r^s \mathcal{H}^s(C) > 0$ which in turn gives the result.

3. Starting from the second condition and using the exact same technique as above we obtain $a^s \mathcal{H}^s(N \cap C) \leq r^s \mathcal{H}(C)$ which gives the result.

□

Example 4.24. *If $f_c = z^2 + c$ for $|c| < 1/4$ then $J(f_c)$ is a quasi-circle. This follows since f_c is a mixing repeller on J (see Definition 4.5 and Lemmas 4.18 and 4.17), it is $C^{1+\alpha}$ for $\alpha = 1$ (See Definition 4.4) and conformal since it is holomorphic and its derivative is everywhere non-zero on $J(f_c)$. Hence by Theorem 4.6 it satisfies conditions 2 and 3 of the Definition of a quasi-circle. By Theorem 4.21 it also satisfies the first condition.*
We can also apply Remark 4.13 to deduce that the Hausdorff Dimension of $J(f_c)$ is analytic in c. Ruelle also calculated explicitly $\dim_{\mathcal{H}}(J(f_c)) = 1 + \frac{|c|^2}{4\log 2} + o(|c|^3)$ using Bowen's Formula (4.12) in [28]. This is an interesting result, recall before that for the iterated function systems of section 3, where we fixed a set of affine contractions T_i and varied the translations a_i, the Hausdorff dimension was not even continuous as a function of the a_i.

Lemma 4.25. *If C is a quasi-circle then there exist constants c_1, c_2 such that the following holds*

$$0 < c_1 \leq \frac{\mathcal{H}^s(I(x,y))}{|x-y|^s} \leq c_2$$

For a proof of this Lemma see [22], its not included here as it doesn't provide any real insight into the area, it simply follows from direct calculations of $\mathcal{H}^s(I(x,y))$. However from this connection between $\mathcal{H}^s(I(x,y))$ and the distance between x and y it is possible to prove the following very powerful statement.

Lemma 4.26. *Let C be a quasi-circle with $s = \dim_{\mathcal{H}} C$ then there exists a bijection $f : C \to S^1$ satisfying*

$$\alpha|x-y|^s \leq \theta(f(x), f(y)) \leq \beta|x-y|^s$$

for some constants α, β and where $\theta(x,y)$ is the smallest angle between x and y on S^1

Proof. Let $m = \mathcal{H}^s(C)$ and recall that by Lemma 4.23 we have that $0 < m < \infty$. Choose a point $p \in C$ and define $f : C \to S^1$ by

$$f(x) = (2\pi/m)\mathcal{H}^s(I(p,x))$$

Notice that f is strictly monotonic, since $x < y$ implies $I(p,x) \subset I(p,y)$ which implies $\mathcal{H}^s(I(p,x)) \leq \mathcal{H}^s(I(p,y))$ and f is continuous by Lemma 4.25. Furthermore $f(0) = 0$ by continuity, $\mathcal{H}^s(I(x,y)) \to 0$ as $y \to x$ (Lemma 4.23) and $f(2\pi) = 2\pi$. Combining the previous three facts, we see that f is bijective.
Since C is compact (homeomorphic image of a compact space) I is uniformly continuous. Hence there exists $\varepsilon > 0$ such that $|x-y| < \varepsilon$ implies $|I(x,y)| < r_0$ and $\theta(f(x), f(y)) < 1/2$ (this final condition is just asking that we take the smallest angle between $f(x)$ and $f(y)$). Now by Lemma 4.25 we know

$$0 < c_1 \leq \frac{\mathcal{H}^s(I(x,y))}{|x-y|^s} \leq c_1$$

for some constants c_1, c_2. Using $\theta(f(x), f(y)) = (2\pi/m)\mathcal{H}^s(I(x,y))$ We can rewrite this as

$$0 < c_1 \leq \frac{m}{2\pi} \frac{\theta(f(x), f(y))}{|x-y|^s} \leq c_2$$

This gives the result. □

We are now ready to prove our main theorem, due to Falconer and Marsh in [22]

Theorem 4.27. *Quasi-circles C_1 and C_2 are Lipschitz equivalent if and only if $\dim_{\mathcal{H}}(C_1) = \dim_{\mathcal{H}}(C_2)$*

Proof. Recalling from Lemma 1.10 that bi-Lipschitz mappings preserve Hausdorff dimension its clear C_1 and C_2 being Lipschitz-equivalent implies that $\dim_{\mathcal{H}}(C_1) = \dim_{\mathcal{H}}(C_2)$ For the opposite inclusion, if $\dim_{\mathcal{H}}(C_1) = \dim_{\mathcal{H}}(C_2)$ then by Lemma 4.26 we get bijections $f_1 : C_1 \to S^1$ and $f_2 : C_2 \to S^1$ which satisfy the holder condition of exponent α. Hence $f_2 \circ f_1^{-1} : C_1 \to C_2$ is a bi-Lipschitz bijection. □

5 Remarks about the Figures

The pictures of self-affine fractals were plotted using the following observations

1. given an iterated function system $\{S_1, \ldots, S_m\}$ with invariant (or sub-self-similar) set Λ, the function $S : K \to \Lambda$ could have been defined as

$$S(\mathbf{i}) = \bigcap_{r=0}^{\infty} S_{i_0} \circ \cdots \circ S_{i_r}(z)$$

for any point $z \in \mathbb{R}^n$.

2. Since we are always dealing with contractions, it is sufficient to only calculate $S_{i_0} \circ \cdots \circ S_{i_r}(z)$ for sufficiently large r, in order to get close to Λ.

These observations can be exploited to easily plot very good approximations of the invariant sets. Take any point $z \in \mathbb{R}^n$ (in practice you want it to be fairly close to Λ) then apply the transformations S_i chosen at random enough times to be very close to Λ and plot that point. After repeating the above enough times you build up a very good image of Λ. I wrote a very small C program using the above techniques, which I can provide to anyone interested.

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